# Entropy, Randomness, and Information

Lecture 27

December 5, 2013

# Part I

# Entropy

# Quote

*"If only once - only once - no matter where, no matter before what audience - I could better the record of the great Rastelli and juggle with thirteen balls, instead of my usual twelve, I would feel that I had truly accomplished something for my country. But I am not getting any younger, and although I am still at the peak of my powers there are moments - why deny it? - when I begin to doubt - and there is a time limit on all of us."*
*–Romain Gary, The talent scout.*

# Entropy: Definition

## Definition

The ***entropy*** in bits of a discrete random variable $X$ is

$$\mathbb{H}(X) = -\sum_{x} \Pr\Big[X = x\Big] \lg \Pr\Big[X = x\Big].$$

Equivalently, $\mathbb{H}(X) = \mathbf{E}\Big[\lg \frac{1}{\Pr[X]}\Big]$.

# Entropy intuition...

## Intuition...

$\mathbb{H}(\boldsymbol{X})$ is the number of fair coin flips that one gets when getting the value of $\boldsymbol{X}$.

# Binary entropy

$$\mathbb{H}(X) = -\sum_x \Pr\Big[X = x\Big] \lg \Pr\Big[X = x\Big]$$

$\Longrightarrow$

**Definition**

The ***binary entropy*** function $\mathbb{H}(p)$ for a random binary variable that is $1$ with probability $p$, is $\mathbb{H}(p) = -p \lg p - (1-p)\lg(1-p)$. We define $\mathbb{H}(0) = \mathbb{H}(1) = 0$.

Q: How many truly random bits are there when given the result of flipping a single coin with probability $p$ for heads?

# Binary entropy

$\mathbb{H}(X) = -\sum_x \Pr\Big[X = x\Big] \lg \Pr\Big[X = x\Big]$
$\Longrightarrow$

## Definition

The ***binary entropy*** function $\mathbb{H}(p)$ for a random binary variable that is $1$ with probability $p$, is $\mathbb{H}(p) = -p \lg p - (1-p)\lg(1-p)$. We define $\mathbb{H}(0) = \mathbb{H}(1) = 0$.

Q: How many truly random bits are there when given the result of flipping a single coin with probability $p$ for heads?

# Binary entropy

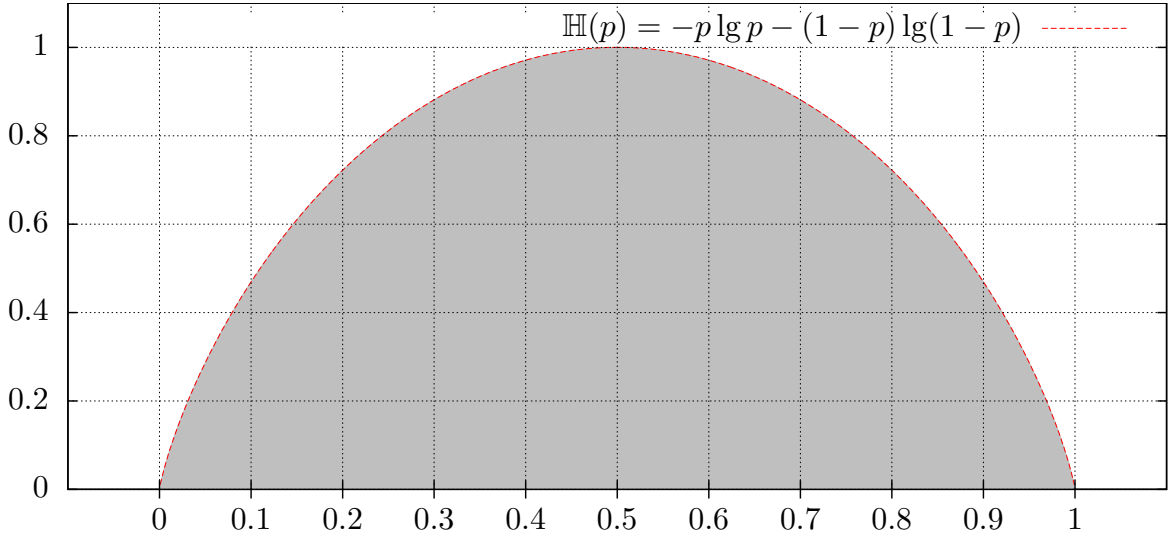
$\mathbb{H}(X) = -\sum_x \Pr\Big[X = x\Big] \lg \Pr\Big[X = x\Big]$
$\Longrightarrow$

## Definition

The ***binary entropy*** function $\mathbb{H}(p)$ for a random binary variable that is $1$ with probability $p$, is $\mathbb{H}(p) = -p \lg p - (1-p)\lg(1-p)$. We define $\mathbb{H}(0) = \mathbb{H}(1) = 0$.

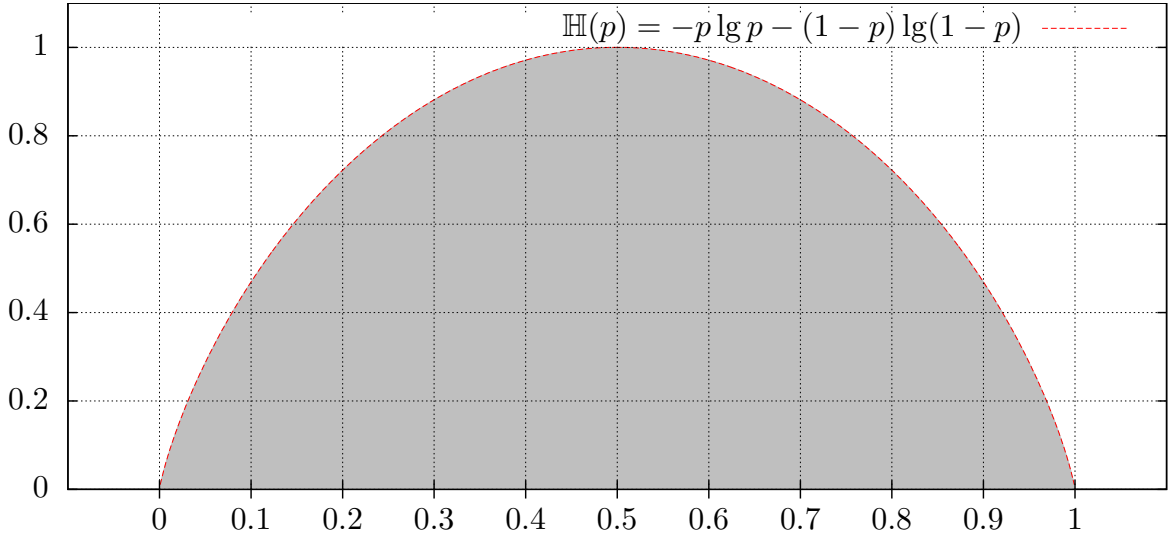
Q: How many truly random bits are there when given the result of flipping a single coin with probability $p$ for heads?

# Binary entropy:
# $\mathbb{H}(\mathrm{p}) = -\mathrm{p}\lg\mathrm{p} - (1-\mathrm{p})\lg(1-\mathrm{p})$



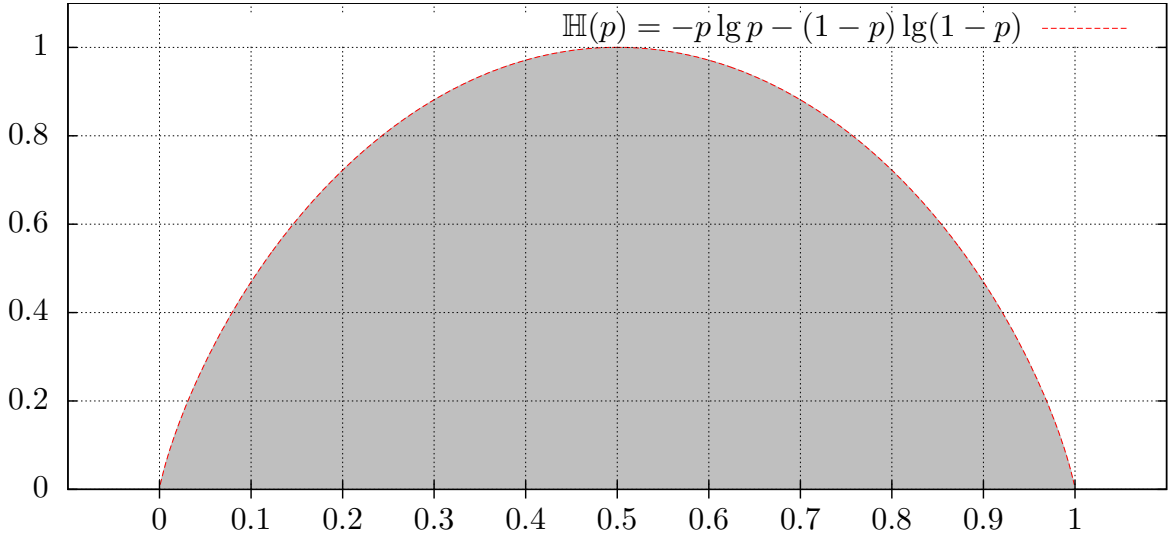

1. $\mathbb{H}(p)$ is a concave symmetric around $1/2$ on the interval $[0, 1]$.
2. maximum at $1/2$.
3. $\mathbb{H}(3/4) \approx 0.8113$ and $\mathbb{H}(7/8) \approx 0.5436$.
4. $\Longrightarrow$ coin that has $3/4$ probably to be heads have higher amount of "randomness" in it than a coin that has probability $7/8$ for heads.

# Binary entropy:
# $\mathbb{H}(\mathrm{p}) = -\mathrm{p}\lg\mathrm{p} - (1-\mathrm{p})\lg(1-\mathrm{p})$



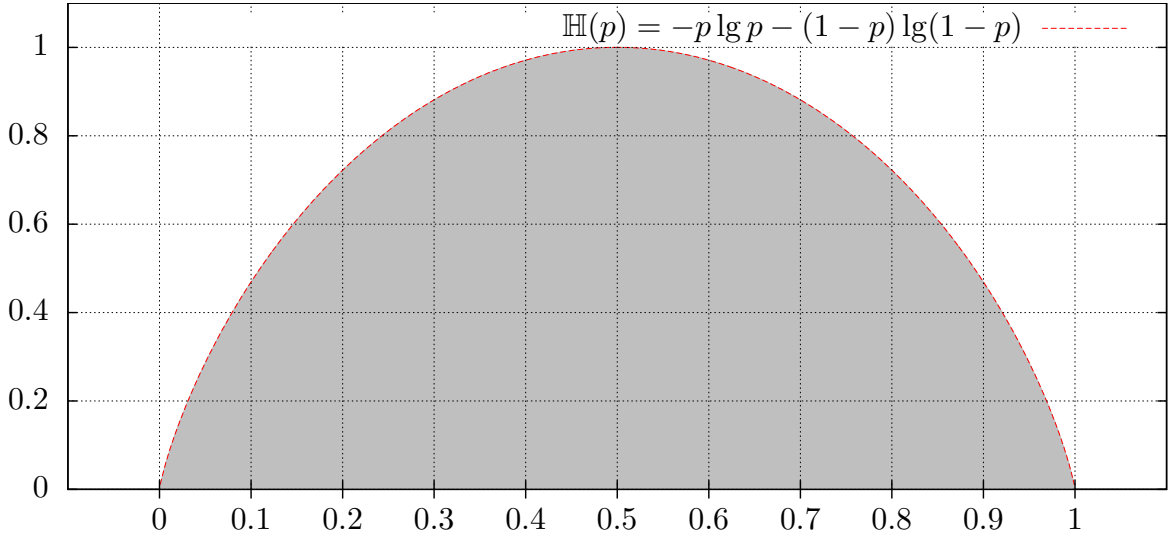

1. $\mathbb{H}(p)$ is a concave symmetric around $1/2$ on the interval $[0, 1]$.
2. maximum at $1/2$.
3. $\mathbb{H}(3/4) \approx 0.8113$ and $\mathbb{H}(7/8) \approx 0.5436$.
4. $\implies$ coin that has $3/4$ probably to be heads have higher amount of "randomness" in it than a coin that has probability $7/8$ for heads.

# Binary entropy:
# $\mathbb{H}(\mathrm{p}) = -\mathrm{p}\lg\mathrm{p} - (1-\mathrm{p})\lg(1-\mathrm{p})$




1. $\mathbb{H}(p)$ is a concave symmetric around $1/2$ on the interval $[0,1]$.
2. maximum at $1/2$.
3. $\mathbb{H}(3/4) \approx 0.8113$ and $\mathbb{H}(7/8) \approx 0.5436$.
4. $\Longrightarrow$ coin that has $3/4$ probably to be heads have higher amount of "randomness" in it than a coin that has probability $7/8$ for heads.

# Binary entropy:
# $\mathbb{H}(\mathrm{p}) = -\mathrm{p}\lg\mathrm{p} - (1-\mathrm{p})\lg(1-\mathrm{p})$




1. $\mathbb{H}(p)$ is a concave symmetric around $1/2$ on the interval $[0, 1]$.
2. maximum at $1/2$.
3. $\mathbb{H}(3/4) \approx 0.8113$ and $\mathbb{H}(7/8) \approx 0.5436$.
4. $\implies$ coin that has $3/4$ probably to be heads have higher amount of "randomness" in it than a coin that has probability $7/8$ for heads.

# And now for some unnecessary math

1. $\mathbb{H}(p) = -p \lg p - (1-p) \lg(1-p)$
2. $\mathbb{H}'(p) = -\lg p + \lg(1-p) = \lg \frac{1-p}{p}$
3. $\mathbb{H}''(p) = \frac{p}{1-p} \cdot \left(-\frac{1}{p^2}\right) = -\frac{1}{p(1-p)}$.
4. $\implies \mathbb{H}''(p) \leq 0$, for all $p \in (0,1)$, and the $\mathbb{H}(\cdot)$ is concave.
5. $\mathbb{H}'(1/2) = 0 \implies \mathbb{H}(1/2) = 1$ **max** of binary entropy.
6. $\implies$ balanced coin has the largest amount of randomness in it.

# And now for some unnecessary math

1. $\mathbb{H}(p) = -p \lg p - (1-p) \lg(1-p)$
2. $\mathbb{H}'(p) = -\lg p + \lg(1-p) = \lg \frac{1-p}{p}$
3. $\mathbb{H}''(p) = \frac{p}{1-p} \cdot \left(-\frac{1}{p^2}\right) = -\frac{1}{p(1-p)}$.
4. $\implies \mathbb{H}''(p) \leq 0$, for all $p \in (0,1)$, and the $\mathbb{H}(\cdot)$ is concave.
5. $\mathbb{H}'(1/2) = 0 \implies \mathbb{H}(1/2) = 1$ **max** of binary entropy.
6. $\implies$ balanced coin has the largest amount of randomness in it.

# And now for some unnecessary math

1. $\mathbb{H}(p) = -p \lg p - (1-p) \lg(1-p)$
2. $\mathbb{H}'(p) = -\lg p + \lg(1-p) = \lg \frac{1-p}{p}$
3. $\mathbb{H}''(p) = \frac{p}{1-p} \cdot \left(-\frac{1}{p^2}\right) = -\frac{1}{p(1-p)}$.
4. $\implies \mathbb{H}''(p) \leq 0$, for all $p \in (0,1)$, and the $\mathbb{H}(\cdot)$ is concave.
5. $\mathbb{H}'(1/2) = 0 \implies \mathbb{H}(1/2) = 1$ **max** of binary entropy.
6. $\implies$ balanced coin has the largest amount of randomness in it.

# And now for some unnecessary math

1. $\mathbb{H}(p) = -p \lg p - (1-p)\lg(1-p)$
2. $\mathbb{H}'(p) = -\lg p + \lg(1-p) = \lg \frac{1-p}{p}$
3. $\mathbb{H}''(p) = \frac{p}{1-p} \cdot \left(-\frac{1}{p^2}\right) = -\frac{1}{p(1-p)}$.
4. $\implies \mathbb{H}''(p) \leq 0$, for all $p \in (0,1)$, and the $\mathbb{H}(\cdot)$ is concave.
5. $\mathbb{H}'(1/2) = 0 \implies \mathbb{H}(1/2) = 1$ **max** of binary entropy.
6. $\implies$ balanced coin has the largest amount of randomness in it.

# And now for some unnecessary math

1. $\mathbb{H}(p) = -p \lg p - (1-p)\lg(1-p)$
2. $\mathbb{H}'(p) = -\lg p + \lg(1-p) = \lg \frac{1-p}{p}$
3. $\mathbb{H}''(p) = \frac{p}{1-p} \cdot \left(-\frac{1}{p^2}\right) = -\frac{1}{p(1-p)}$.
4. $\implies \mathbb{H}''(p) \leq 0$, for all $p \in (0,1)$, and the $\mathbb{H}(\cdot)$ is concave.
5. $\mathbb{H}'(1/2) = 0 \implies \mathbb{H}(1/2) = 1$ **max** of binary entropy.
6. $\implies$ balanced coin has the largest amount of randomness in it.

# And now for some unnecessary math

1. $\mathbb{H}(p) = -p \lg p - (1-p) \lg(1-p)$
2. $\mathbb{H}'(p) = -\lg p + \lg(1-p) = \lg \frac{1-p}{p}$
3. $\mathbb{H}''(p) = \frac{p}{1-p} \cdot \left(-\frac{1}{p^2}\right) = -\frac{1}{p(1-p)}$.
4. $\implies \mathbb{H}''(p) \leq 0$, for all $p \in (0,1)$, and the $\mathbb{H}(\cdot)$ is concave.
5. $\mathbb{H}'(1/2) = 0 \implies \mathbb{H}(1/2) = 1$ **max** of binary entropy.
6. $\implies$ balanced coin has the largest amount of randomness in it.

# Squeezing good random bits out of bad random bits...

Given the result of $n$ coin flips: $b_1, \ldots, b_n$ from a faulty coin, with head with probability $p$, how many truly random bits can we extract?

# Squeezing good random bits out of bad random bits...

## Question...

Given the result of $n$ coin flips: $b_1, \ldots, b_n$ from a faulty coin, with head with probability $p$, how many truly random bits can we extract?

If believe intuition about entropy, then this number should be $\approx n\mathbb{H}(p)$.

# Back to Entropy

1. ***entropy*** of $X$ is $\mathbb{H}(X) = -\sum_x \Pr\left[X = x\right] \lg \Pr\left[X = x\right]$.
2. Entropy of uniform variable..

**Example**

A random variable $X$ that has probability $1/n$ to be $i$, for $i = 1, \ldots, n$, has entropy $\mathbb{H}(X) = -\sum_{i=1}^{n} \frac{1}{n} \lg \frac{1}{n} = \lg n$.

3. Entropy is oblivious to the exact values random variable can have.
4. $\implies$ random variables over $-1, +1$ with equal probability has the same entropy (i.e., $1$) as a fair coin.

# Back to Entropy

1. ***entropy*** of $X$ is $\mathbb{H}(X) = -\sum_x \Pr\left[X = x\right] \lg \Pr\left[X = x\right]$.
2. Entropy of uniform variable..

**Example**

A random variable $X$ that has probability $1/n$ to be $i$, for $i = 1, \ldots, n$, has entropy $\mathbb{H}(X) = -\sum_{i=1}^{n} \frac{1}{n} \lg \frac{1}{n} = \lg n$.

3. Entropy is oblivious to the exact values random variable can have.
4. $\implies$ random variables over $-1, +1$ with equal probability has the same entropy (i.e., $1$) as a fair coin.

# Back to Entropy

1. ***entropy*** of $X$ is $\mathbb{H}(X) = -\sum_x \Pr\left[X = x\right] \lg \Pr\left[X = x\right]$.
2. Entropy of uniform variable..

**Example**

A random variable $X$ that has probability $1/n$ to be $i$, for $i = 1, \ldots, n$, has entropy $\mathbb{H}(X) = -\sum_{i=1}^{n} \frac{1}{n} \lg \frac{1}{n} = \lg n$.

3. Entropy is oblivious to the exact values random variable can have.
4. $\implies$ random variables over $-1, +1$ with equal probability has the same entropy (i.e., $1$) as a fair coin.

# Back to Entropy

1. ***entropy*** of $X$ is $\mathbb{H}(X) = -\sum_x \Pr\left[X = x\right] \lg \Pr\left[X = x\right]$.
2. Entropy of uniform variable..

**Example**

A random variable $X$ that has probability $1/n$ to be $i$, for $i = 1, \ldots, n$, has entropy $\mathbb{H}(X) = -\sum_{i=1}^{n} \frac{1}{n} \lg \frac{1}{n} = \lg n$.

3. Entropy is oblivious to the exact values random variable can have.
4. $\implies$ random variables over $-1, +1$ with equal probability has the same entropy (i.e., $1$) as a fair coin.

# Back to Entropy

1. ***entropy*** of $X$ is $\mathbb{H}(X) = -\sum_x \Pr\left[X = x\right] \lg \Pr\left[X = x\right]$.
2. Entropy of uniform variable..

**Example**

A random variable $X$ that has probability $1/n$ to be $i$, for $i = 1, \ldots, n$, has entropy $\mathbb{H}(X) = -\sum_{i=1}^{n} \frac{1}{n} \lg \frac{1}{n} = \lg n$.

3. Entropy is oblivious to the exact values random variable can have.
4. $\implies$ random variables over $-1, +1$ with equal probability has the same entropy (i.e., $1$) as a fair coin.

## Lemma

*Let* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *be two independent random variables, and let* $\boldsymbol{Z}$ *be the random variable* $(\boldsymbol{X}, \boldsymbol{Y})$*. Then* $\mathbb{H}(\boldsymbol{Z}) = \mathbb{H}(\boldsymbol{X}) + \mathbb{H}(\boldsymbol{Y})$*.*

# Proof

In the following, summation are over all possible values that the variables can have. By the independence of $X$ and $Y$ we have

$$\begin{aligned}
\mathbb{H}(Z) &= \sum_{x,y} \Pr\Big[(X, Y) = (x, y)\Big] \lg \frac{1}{\Pr[(X, Y) = (x, y)]} \\
&= \sum_{x,y} \Pr\Big[X = x\Big] \Pr\Big[Y = y\Big] \lg \frac{1}{\Pr[X = x] \Pr[Y = y]} \\
&= \sum_{x} \sum_{y} \Pr[X = x] \Pr[Y = y] \lg \frac{1}{\Pr[X = x]} \\
&\quad + \sum_{y} \sum_{x} \Pr[X = x] \Pr[Y = y] \lg \frac{1}{\Pr[Y = y]}
\end{aligned}$$

# Proof continued

$$\begin{aligned}
\mathbb{H}(Z) &= \sum_x \sum_y \Pr[X = x] \Pr[Y = y] \lg \frac{1}{\Pr[X = x]} \\
&\quad + \sum_y \sum_x \Pr[X = x] \Pr[Y = y] \lg \frac{1}{\Pr[Y = y]} \\
&= \sum_x \Pr[X = x] \lg \frac{1}{\Pr[X = x]} \\
&\quad + \sum_y \Pr[Y = y] \lg \frac{1}{\Pr[Y = y]} \\
&= \mathbb{H}(X) + \mathbb{H}(Y) .
\end{aligned}$$

■

# Bounding the binomial coefficient using entropy

## Lemma

*Suppose that* $nq$ *is integer in the range* $[0, n]$*. Then*
$$\frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{nq} \leq 2^{n\mathbb{H}(q)}.$$

# Proof

Holds if $q = 0$ or $q = 1$, so assume $0 < q < 1$. We have

$$\binom{n}{nq} q^{nq}(1-q)^{n-nq} \leq (q + (1-q))^n = 1.$$

As such, since $q^{-nq}(1-q)^{-(1-q)n} = 2^{n(-q \lg q - (1-q)\lg(1-q))} = 2^{n\mathbb{H}(q)}$, we have

$$\binom{n}{nq} \leq q^{-nq}(1-q)^{-(1-q)n} = 2^{n\mathbb{H}(q)}.$$

# Proof

Holds if $q = 0$ or $q = 1$, so assume $0 < q < 1$. We have

$$\binom{n}{nq} q^{nq}(1-q)^{n-nq} \leq (q + (1-q))^n = 1.$$

As such, since $q^{-nq}(1-q)^{-(1-q)n} = 2^{n(-q \lg q - (1-q)\lg(1-q))} = 2^{n\mathbb{H}(q)}$, we have

$$\binom{n}{nq} \leq q^{-nq}(1-q)^{-(1-q)n} = 2^{n\mathbb{H}(q)}.$$

# Proof

Holds if $q = 0$ or $q = 1$, so assume $0 < q < 1$. We have

$$\binom{n}{nq} q^{nq}(1-q)^{n-nq} \leq (q + (1-q))^n = 1.$$

As such, since
$q^{-nq}(1-q)^{-(1-q)n} = 2^{n(-q \lg q - (1-q)\lg(1-q))} = 2^{n\mathbb{H}(q)}$, we have

$$\binom{n}{nq} \leq q^{-nq}(1-q)^{-(1-q)n} = 2^{n\mathbb{H}(q)}.$$

# Proof

Holds if $q = 0$ or $q = 1$, so assume $0 < q < 1$. We have

$$\binom{n}{nq} q^{nq}(1-q)^{n-nq} \leq (q + (1-q))^n = 1.$$

As such, since $q^{-nq}(1-q)^{-(1-q)n} = 2^{n(-q \lg q - (1-q)\lg(1-q))} = 2^{n\mathbb{H}(q)}$, we have

$$\binom{n}{nq} \leq q^{-nq}(1-q)^{-(1-q)n} = 2^{n\mathbb{H}(q)}.$$

# Proof continued

## Other direction...

1. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
2. $\sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} = \sum_{i=0}^{n} \mu(i)$.
3. Claim: $\mu(nq) = \binom{n}{nq} q^{nq} (1-q)^{n-nq}$ largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
4. $\Delta_k = \mu(k) - \mu(k+1) = \binom{n}{k} q^k (1-q)^{n-k} \left(1 - \frac{n-k}{k+1} \frac{q}{1-q}\right)$,
5. sign of $\Delta_k$ = size of last term...
6. $\mathrm{sign}(\Delta_k) = \mathrm{sign}\left(1 - \frac{(n-k)q}{(k+1)(1-q)}\right)$
$= \mathrm{sign}\left(\frac{(k+1)(1-q)-(n-k)q}{(k+1)(1-q)}\right)$.

# Proof continued

## Other direction...

1. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
2. $\sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} = \sum_{i=0}^{n} \mu(i)$.
3. Claim: $\mu(nq) = \binom{n}{nq} q^{nq} (1-q)^{n-nq}$ largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
4. $\Delta_k = \mu(k) - \mu(k+1) = \binom{n}{k} q^k (1-q)^{n-k} \left(1 - \frac{n-k}{k+1} \frac{q}{1-q}\right)$,
5. sign of $\Delta_k$ = size of last term...
6. $\mathrm{sign}(\Delta_k) = \mathrm{sign}\left(1 - \frac{(n-k)q}{(k+1)(1-q)}\right)$
$= \mathrm{sign}\left(\frac{(k+1)(1-q)-(n-k)q}{(k+1)(1-q)}\right)$.

# Proof continued

## Other direction...

1. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
2. $\sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} = \sum_{i=0}^{n} \mu(i)$.
3. Claim: $\mu(nq) = \binom{n}{nq} q^{nq} (1-q)^{n-nq}$ largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
4. $\Delta_k = \mu(k) - \mu(k+1) = \binom{n}{k} q^k (1-q)^{n-k} \left(1 - \frac{n-k}{k+1} \frac{q}{1-q}\right)$,
5. sign of $\Delta_k$ = size of last term...
6. $\mathrm{sign}(\Delta_k) = \mathrm{sign}\left(1 - \frac{(n-k)q}{(k+1)(1-q)}\right)$
$= \mathrm{sign}\left(\frac{(k+1)(1-q)-(n-k)q}{(k+1)(1-q)}\right)$.

# Proof continued

## Other direction...

1. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
2. $\sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} = \sum_{i=0}^{n} \mu(i)$.
3. Claim: $\mu(nq) = \binom{n}{nq} q^{nq} (1-q)^{n-nq}$ largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
4. $\Delta_k = \mu(k) - \mu(k+1) = \binom{n}{k} q^k (1-q)^{n-k} \left(1 - \frac{n-k}{k+1} \frac{q}{1-q}\right)$,
5. sign of $\Delta_k$ = size of last term...
6. $\operatorname{sign}(\Delta_k) = \operatorname{sign}\left(1 - \frac{(n-k)q}{(k+1)(1-q)}\right)$
$= \operatorname{sign}\left(\frac{(k+1)(1-q)-(n-k)q}{(k+1)(1-q)}\right)$.

# Proof continued

## Other direction...

1. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
2. $\sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} = \sum_{i=0}^{n} \mu(i)$.
3. Claim: $\mu(nq) = \binom{n}{nq} q^{nq} (1-q)^{n-nq}$ largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
4. $\Delta_k = \mu(k) - \mu(k+1) = \binom{n}{k} q^k (1-q)^{n-k} \left(1 - \frac{n-k}{k+1} \frac{q}{1-q}\right)$,
5. sign of $\Delta_k$ = size of last term...
6. $\operatorname{sign}(\Delta_k) = \operatorname{sign}\left(1 - \frac{(n-k)q}{(k+1)(1-q)}\right)$
$= \operatorname{sign}\left(\frac{(k+1)(1-q)-(n-k)q}{(k+1)(1-q)}\right)$.

# Proof continued

## Other direction...

1. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
2. $\sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} = \sum_{i=0}^{n} \mu(i)$.
3. Claim: $\mu(nq) = \binom{n}{nq} q^{nq} (1-q)^{n-nq}$ largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
4. $\Delta_k = \mu(k) - \mu(k+1) = \binom{n}{k} q^k (1-q)^{n-k} \left(1 - \frac{n-k}{k+1} \frac{q}{1-q}\right)$,
5. sign of $\Delta_k$ = size of last term...
6. $\mathrm{sign}(\Delta_k) = \mathrm{sign}\left(1 - \frac{(n-k)q}{(k+1)(1-q)}\right)$
$$= \mathrm{sign}\left(\frac{(k+1)(1-q)-(n-k)q}{(k+1)(1-q)}\right).$$

# Proof continued

## Other direction...

1. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
2. $\sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} = \sum_{i=0}^{n} \mu(i)$.
3. Claim: $\mu(nq) = \binom{n}{nq} q^{nq} (1-q)^{n-nq}$ largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
4. $\Delta_k = \mu(k) - \mu(k+1) = \binom{n}{k} q^k (1-q)^{n-k} \left(1 - \frac{n-k}{k+1} \frac{q}{1-q}\right)$,
5. sign of $\Delta_k$ = size of last term...
6. $\mathrm{sign}(\Delta_k) = \mathrm{sign}\left(1 - \frac{(n-k)q}{(k+1)(1-q)}\right)$
$$= \mathrm{sign}\left(\frac{(k+1)(1-q)-(n-k)q}{(k+1)(1-q)}\right).$$

# Proof continued

1. $(k+1)(1-q)-(n-k)q = k+1-kq-q-nq+kq = 1+k-q-nq$.
2. $\implies \Delta_k \geq 0$ when $k \geq nq+q-1$ $\Delta_k < 0$ otherwise.
3. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
4. $\mu(k) < \mu(k+1)$, for $k < nq$, and $\mu(k) \geq \mu(k+1)$ for $k \geq nq$.
5. $\implies \mu(nq)$ is the largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
6. $\mu(nq)$ larger than the average in sum.
7. $\implies \binom{n}{k} q^k (1-q)^{n-k} \geq \frac{1}{n+1}$.
8. $\implies \binom{n}{nq} \geq \frac{1}{n+1} q^{-nq} (1-q)^{-(n-nq)} = \frac{1}{n+1} 2^{n\mathbb{H}(q)}$. ∎

# Proof continued

1. $(k+1)(1-q)-(n-k)q =$
   $k+1-kq-q-nq+kq = 1+k-q-nq$.
2. $\implies \Delta_k \geq 0$ when $k \geq nq+q-1$
   $\Delta_k < 0$ otherwise.
3. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
4. $\mu(k) < \mu(k+1)$, for $k < nq$, and $\mu(k) \geq \mu(k+1)$ for $k \geq nq$.
5. $\implies \mu(nq)$ is the largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
6. $\mu(nq)$ larger than the average in sum.
7. $\implies \binom{n}{k} q^k (1-q)^{n-k} \geq \frac{1}{n+1}$.
8. $\implies \binom{n}{nq} \geq \frac{1}{n+1} q^{-nq} (1-q)^{-(n-nq)} = \frac{1}{n+1} 2^{n\mathbb{H}(q)}$. ∎

# Proof continued

1. $(k+1)(1-q)-(n-k)q =$
   $k+1-kq-q-nq+kq = 1+k-q-nq$.
2. $\implies \Delta_k \geq 0$ when $k \geq nq+q-1$
   $\Delta_k < 0$ otherwise.
3. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
4. $\mu(k) < \mu(k+1)$, for $k < nq$, and $\mu(k) \geq \mu(k+1)$ for $k \geq nq$.
5. $\implies \mu(nq)$ is the largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
6. $\mu(nq)$ larger than the average in sum.
7. $\implies \binom{n}{k} q^k (1-q)^{n-k} \geq \frac{1}{n+1}$.
8. $\implies \binom{n}{nq} \geq \frac{1}{n+1} q^{-nq} (1-q)^{-(n-nq)} = \frac{1}{n+1} 2^{n\mathbb{H}(q)}$. ∎

# Proof continued

1. $(k+1)(1-q)-(n-k)q =$
   $k+1-kq-q-nq+kq = 1+k-q-nq$.
2. $\implies \Delta_k \geq 0$ when $k \geq nq+q-1$
   $\Delta_k < 0$ otherwise.
3. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
4. $\mu(k) < \mu(k+1)$, for $k < nq$, and $\mu(k) \geq \mu(k+1)$ for $k \geq nq$.
5. $\implies \mu(nq)$ is the largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
6. $\mu(nq)$ larger than the average in sum.
7. $\implies \binom{n}{k} q^k (1-q)^{n-k} \geq \frac{1}{n+1}$.
8. $\implies \binom{n}{nq} \geq \frac{1}{n+1} q^{-nq} (1-q)^{-(n-nq)} = \frac{1}{n+1} 2^{n\mathbb{H}(q)}$. ∎

# Proof continued

1. $(k+1)(1-q)-(n-k)q =$
   $k+1-kq-q-nq+kq = 1+k-q-nq$.
2. $\implies \Delta_k \geq 0$ when $k \geq nq+q-1$
   $\Delta_k < 0$ otherwise.
3. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
4. $\mu(k) < \mu(k+1)$, for $k < nq$, and $\mu(k) \geq \mu(k+1)$ for $k \geq nq$.
5. $\implies \mu(nq)$ is the largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
6. $\mu(nq)$ larger than the average in sum.
7. $\implies \binom{n}{k} q^k (1-q)^{n-k} \geq \frac{1}{n+1}$.
8. $\implies \binom{n}{nq} \geq \frac{1}{n+1} q^{-nq} (1-q)^{-(n-nq)} = \frac{1}{n+1} 2^{n\mathbb{H}(q)}$. ∎

# Proof continued

1. $(k+1)(1-q)-(n-k)q =$
   $k+1-kq-q-nq+kq = 1+k-q-nq$.
2. $\implies \Delta_k \geq 0$ when $k \geq nq+q-1$
   $\Delta_k < 0$ otherwise.
3. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
4. $\mu(k) < \mu(k+1)$, for $k < nq$, and $\mu(k) \geq \mu(k+1)$ for $k \geq nq$.
5. $\implies \mu(nq)$ is the largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
6. $\mu(nq)$ larger than the average in sum.
7. $\implies \binom{n}{k} q^k (1-q)^{n-k} \geq \frac{1}{n+1}$.
8. $\implies \binom{n}{nq} \geq \frac{1}{n+1} q^{-nq} (1-q)^{-(n-nq)} = \frac{1}{n+1} 2^{n\mathbb{H}(q)}$. ∎

# Proof continued

1. $(k+1)(1-q)-(n-k)q =$
   $k+1-kq-q-nq+kq = 1+k-q-nq$.
2. $\implies \Delta_k \geq 0$ when $k \geq nq+q-1$
   $\Delta_k < 0$ otherwise.
3. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
4. $\mu(k) < \mu(k+1)$, for $k < nq$, and $\mu(k) \geq \mu(k+1)$ for $k \geq nq$.
5. $\implies \mu(nq)$ is the largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
6. $\mu(nq)$ larger than the average in sum.
7. $\implies \binom{n}{k} q^k (1-q)^{n-k} \geq \frac{1}{n+1}$.
8. $\implies \binom{n}{nq} \geq \frac{1}{n+1} q^{-nq} (1-q)^{-(n-nq)} = \frac{1}{n+1} 2^{n\mathbb{H}(q)}$. ∎

# Proof continued

1. $(k+1)(1-q)-(n-k)q =$
   $k+1-kq-q-nq+kq = 1+k-q-nq$.
2. $\implies \Delta_k \geq 0$ when $k \geq nq+q-1$
   $\Delta_k < 0$ otherwise.
3. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
4. $\mu(k) < \mu(k+1)$, for $k < nq$, and $\mu(k) \geq \mu(k+1)$ for $k \geq nq$.
5. $\implies \mu(nq)$ is the largest term in $\sum_{k=0}^{n} \mu(k) = 1$.
6. $\mu(nq)$ larger than the average in sum.
7. $\implies \binom{n}{k} q^k (1-q)^{n-k} \geq \frac{1}{n+1}$.
8. $\implies \binom{n}{nq} \geq \frac{1}{n+1} q^{-nq} (1-q)^{-(n-nq)} = \frac{1}{n+1} 2^{n\mathbb{H}(q)}$. ■

# Generalization...

## Corollary

*We have:*

*(i)* $q \in [0, 1/2] \Rightarrow \binom{n}{\lfloor nq \rfloor} \leq 2^{n\mathbb{H}(q)}$.

*(ii)* $q \in [1/2, 1]$ $\binom{n}{\lceil nq \rceil} \leq 2^{n\mathbb{H}(q)}$.

*(iii)* $q \in [1/2, 1] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lfloor nq \rfloor}$.

*(iv)* $q \in [0, 1/2] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lceil nq \rceil}$.

Proof is straightforward but tedious.

# What we have...

1. Proved that $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$.
2. Estimate is loose.
3. Sanity check...
   (I) A sequence of $n$ bits generated by coin with probability $q$ for head.
   (II) By Chernoff inequality... roughly $nq$ heads in this sequence.
   (III) Generated sequence $Y$ belongs to $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$ possible sequences .
   (IV) ...of similar probability.
   (V) $\implies \mathbb{H}(Y) \approx \lg \binom{n}{nq} = n\mathbb{H}(q)$.

# What we have...

1. Proved that $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$.
2. Estimate is loose.
3. Sanity check...
   (I) A sequence of $n$ bits generated by coin with probability $q$ for head.
   (II) By Chernoff inequality... roughly $nq$ heads in this sequence.
   (III) Generated sequence $Y$ belongs to $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$ possible sequences .
   (IV) ...of similar probability.
   (V) $\implies \mathbb{H}(Y) \approx \lg \binom{n}{nq} = n\mathbb{H}(q)$.

# What we have...

1. Proved that $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$.
2. Estimate is loose.
3. Sanity check...
   (I) A sequence of $n$ bits generated by coin with probability $q$ for head.
   (II) By Chernoff inequality... roughly $nq$ heads in this sequence.
   (III) Generated sequence $Y$ belongs to $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$ possible sequences .
   (IV) ...of similar probability.
   (V) $\implies \mathbb{H}(Y) \approx \lg \binom{n}{nq} = n\mathbb{H}(q)$.

# What we have...

1. Proved that $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$.
2. Estimate is loose.
3. Sanity check...
   (I) A sequence of $n$ bits generated by coin with probability $q$ for head.
   (II) By Chernoff inequality... roughly $nq$ heads in this sequence.
   (III) Generated sequence $Y$ belongs to $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$ possible sequences .
   (IV) ...of similar probability.
   (V) $\implies \mathbb{H}(Y) \approx \lg \binom{n}{nq} = n\mathbb{H}(q)$.

# What we have...

1. Proved that $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$.
2. Estimate is loose.
3. Sanity check...
   (I) A sequence of $n$ bits generated by coin with probability $q$ for head.
   (II) By Chernoff inequality... roughly $nq$ heads in this sequence.
   (III) Generated sequence $Y$ belongs to $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$ possible sequences .
   (IV) ...of similar probability.
   (V) $\implies \mathbb{H}(Y) \approx \lg \binom{n}{nq} = n\mathbb{H}(q)$.

# Extracting randomness...

Entropy can be interpreted as the amount of unbiased random coin flips can be extracted from a random variable.

## Definition

An extraction function **Ext** takes as input the value of a random variable $X$ and outputs a sequence of bits $y$, such that $\Pr\left[\mathsf{Ext}(X) = y \;\middle|\; |y| = k\right] = \frac{1}{2^k}$, whenever $\Pr[|y| = k] > 0$, where $|y|$ denotes the length of $y$.

# Extracting randomness...

1. $X$: uniform random integer variable out of $0, \ldots, 7$.
2. $\mathsf{Ext}(X)$: binary representation of $x$.
3. Definition more subtle... all extracted sequence of the same length would have the same probability.
4. $X$: uniform random integer variable $0, \ldots, 11$.
5. $\mathsf{Ext}(x)$: output the binary representation for $x$ if $0 \leq x \leq 7$.
6. If $x$ is between $8$ and $11$?
7. Idea... Output binary representation of $x - 8$ as a two bit number.
8. A valid extractor...
   $\Pr\left[\mathsf{Ext}(X) = 00 \;\middle|\; |\mathsf{Ext}(X)| = 2\right] = \frac{1}{4}$,

# Extracting randomness...

1. $X$: uniform random integer variable out of $0, \ldots, 7$.
2. $\mathsf{Ext}(X)$: binary representation of $x$.
3. Definition more subtle... all extracted sequence of the same length would have the same probability.
4. $X$: uniform random integer variable $0, \ldots, 11$.
5. $\mathsf{Ext}(x)$: output the binary representation for $x$ if $0 \leq x \leq 7$.
6. If $x$ is between $8$ and $11$?
7. Idea... Output binary representation of $x - 8$ as a two bit number.
8. A valid extractor...
   $\Pr\left[\mathsf{Ext}(X) = 00 \;\middle|\; |\mathsf{Ext}(X)| = 2\right] = \frac{1}{4}$,

# Extracting randomness...

1. $X$: uniform random integer variable out of $0, \ldots, 7$.
2. $\mathsf{Ext}(X)$: binary representation of $x$.
3. Definition more subtle... all extracted sequence of the same length would have the same probability.
4. $X$: uniform random integer variable $0, \ldots, 11$.
5. $\mathsf{Ext}(x)$: output the binary representation for $x$ if $0 \leq x \leq 7$.
6. If $x$ is between $8$ and $11$?
7. Idea... Output binary representation of $x - 8$ as a two bit number.
8. A valid extractor...
   $\Pr\left[\mathsf{Ext}(X) = 00 \;\middle|\; |\mathsf{Ext}(X)| = 2\right] = \frac{1}{4}$,

# Extracting randomness...

1. $X$: uniform random integer variable out of $0, \ldots, 7$.
2. $\mathsf{Ext}(X)$: binary representation of $x$.
3. Definition more subtle... all extracted sequence of the same length would have the same probability.
4. $X$: uniform random integer variable $0, \ldots, 11$.
5. $\mathsf{Ext}(x)$: output the binary representation for $x$ if $0 \leq x \leq 7$.
6. If $x$ is between $8$ and $11$?
7. Idea... Output binary representation of $x - 8$ as a two bit number.
8. A valid extractor...
   $\Pr\left[\mathsf{Ext}(X) = 00 \;\middle|\; |\mathsf{Ext}(X)| = 2\right] = \frac{1}{4}$,

# Extracting randomness...

1. $X$: uniform random integer variable out of $0, \ldots, 7$.
2. $\mathsf{Ext}(X)$: binary representation of $x$.
3. Definition more subtle... all extracted sequence of the same length would have the same probability.
4. $X$: uniform random integer variable $0, \ldots, 11$.
5. $\mathsf{Ext}(x)$: output the binary representation for $x$ if $0 \leq x \leq 7$.
6. If $x$ is between $8$ and $11$?
7. Idea... Output binary representation of $x - 8$ as a two bit number.
8. A valid extractor...
   $\Pr\left[\mathsf{Ext}(X) = 00 \;\middle|\; |\mathsf{Ext}(X)| = 2\right] = \frac{1}{4}$,

# Extracting randomness...

1. $X$: uniform random integer variable out of $0, \ldots, 7$.
2. $\mathsf{Ext}(X)$: binary representation of $x$.
3. Definition more subtle... all extracted sequence of the same length would have the same probability.
4. $X$: uniform random integer variable $0, \ldots, 11$.
5. $\mathsf{Ext}(x)$: output the binary representation for $x$ if $0 \leq x \leq 7$.
6. If $x$ is between $8$ and $11$?
7. Idea... Output binary representation of $x - 8$ as a two bit number.
8. A valid extractor...
   $\Pr\left[\mathsf{Ext}(X) = 00 \;\middle|\; |\mathsf{Ext}(X)| = 2\right] = \frac{1}{4}$,

# Extracting randomness...

1. $X$: uniform random integer variable out of $0, \ldots, 7$.
2. $\mathsf{Ext}(X)$: binary representation of $x$.
3. Definition more subtle... all extracted sequence of the same length would have the same probability.
4. $X$: uniform random integer variable $0, \ldots, 11$.
5. $\mathsf{Ext}(x)$: output the binary representation for $x$ if $0 \leq x \leq 7$.
6. If $x$ is between $8$ and $11$?
7. Idea... Output binary representation of $x - 8$ as a two bit number.
8. A valid extractor...
   $\Pr\left[\mathsf{Ext}(X) = 00 \;\middle|\; |\mathsf{Ext}(X)| = 2\right] = \frac{1}{4}$,

# Extracting randomness...

1. $X$: uniform random integer variable out of $0, \ldots, 7$.
2. $\mathsf{Ext}(X)$: binary representation of $x$.
3. Definition more subtle... all extracted sequence of the same length would have the same probability.
4. $X$: uniform random integer variable $0, \ldots, 11$.
5. $\mathsf{Ext}(x)$: output the binary representation for $x$ if $0 \leq x \leq 7$.
6. If $x$ is between $8$ and $11$?
7. Idea... Output binary representation of $x - 8$ as a two bit number.
8. A valid extractor...
   $\Pr\left[\mathsf{Ext}(X) = 00 \;\middle|\; |\mathsf{Ext}(X)| = 2\right] = \frac{1}{4}$,

# Technical lemma

The following is obvious, but we provide a proof anyway.

## Lemma

*Let $x/y$ be a faction, such that $x/y < 1$. Then, for any $i$, we have $x/y < (x+i)/(y+i)$.*

## Proof.

We need to prove that $x(y+i) - (x+i)y < 0$. The left size is equal to $i(x-y)$, but since $y > x$ (as $x/y < 1$), this quantity is negative, as required. $\square$

# A uniform variable extractor...

**Theorem**

*Suppose that the value of a random variable* $X$ *is chosen uniformly at random from the integers* $\{0, \ldots, m-1\}$*. Then there is an extraction function for* $X$ *that outputs on average at least* $\lfloor \lg m \rfloor - 1 = \lfloor \mathbb{H}(X) \rfloor - 1$ *independent and unbiased bits.*

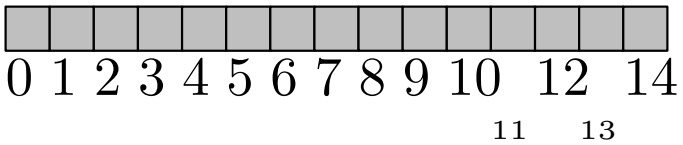
# Proof

1. $m$: A sum of unique powers of $2$, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.
2. Example:
3. decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of $2$.
4. If $x$ is in block $2^k$, output its relative location in the block in binary representation.
5. Example: $x = 10$:
   then falls into block $2^2$...
   $x$ relative location is 2. Output $2$ written using two bits,
   Output: "10".

# Proof

1. $m$: A sum of unique powers of $2$, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.

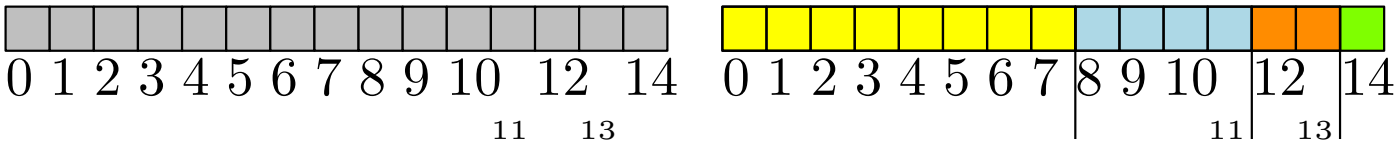
2. Example:




3. decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of $2$.
4. If $x$ is in block $2^k$, output its relative location in the block in binary representation.
5. Example: $x = 10$:
   then falls into block $2^2$...
   $x$ relative location is 2. Output $2$ written using two bits,
   Output: "10".

# Proof

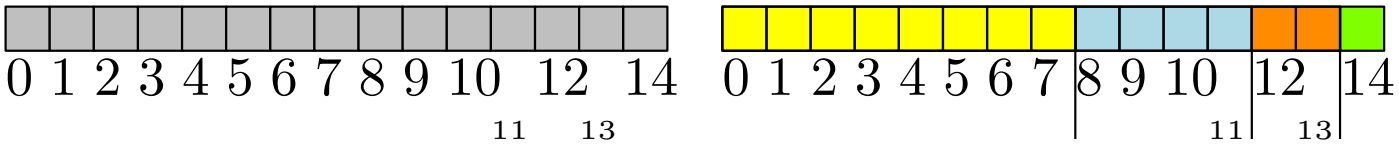
1. $m$: A sum of unique powers of $2$, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.
2. Example:




3. decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of $2$.
4. If $x$ is in block $2^k$, output its relative location in the block in binary representation.
5. Example: $x = 10$:
   then falls into block $2^2$...
   $x$ relative location is 2. Output $2$ written using two bits,
   Output: "10".

# Proof

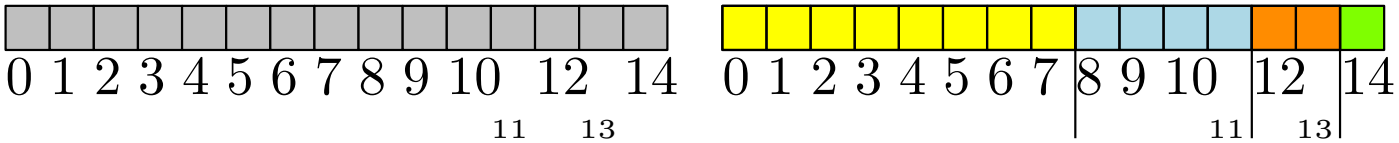
1. $m$: A sum of unique powers of $2$, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.
2. Example:



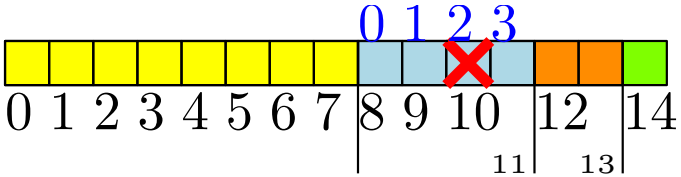

3. decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of $2$.
4. If $x$ is in block $2^k$, output its relative location in the block in binary representation.
5. Example: $x = 10$:
   then falls into block $2^2$...
   $x$ relative location is 2. Output $2$ written using two bits,
   Output: "10".

# Proof

1. $m$: A sum of unique powers of $2$, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.
2. Example:



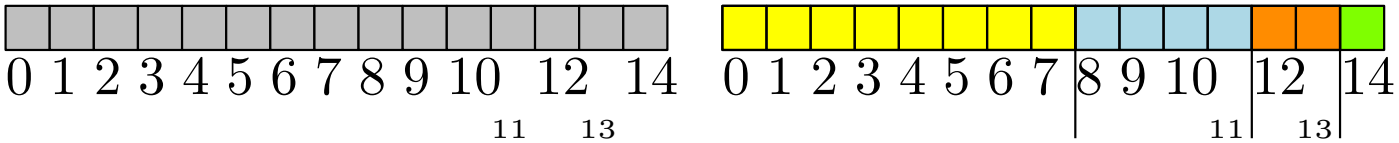

3. decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of $2$.
4. If $x$ is in block $2^k$, output its relative location in the block in binary representation.
5. Example: $x = 10$:



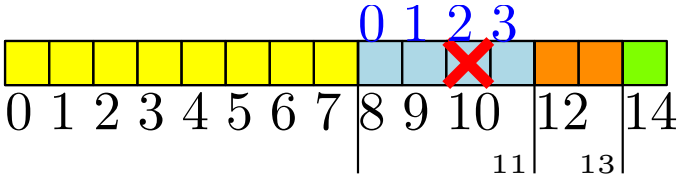

then falls into block $2^2$...
$x$ relative location is 2. Output $2$ written using two bits,
Output: "10".

# Proof

1. $m$: A sum of unique powers of $2$, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.
2. Example:

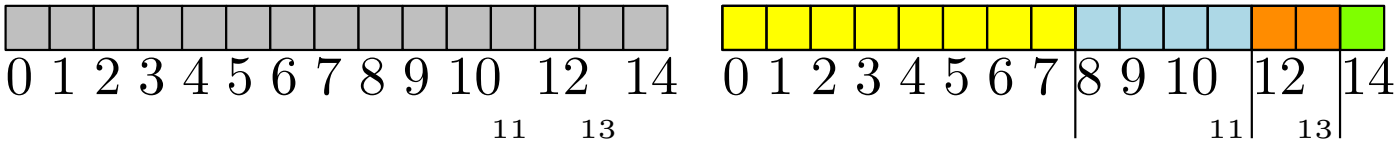



3. decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of $2$.
4. If $x$ is in block $2^k$, output its relative location in the block in binary representation.
5. Example: $x = 10$:
   then falls into block $2^2$...
   $x$ relative location is 2. Output $2$ written using two bits,
   Output: "10".

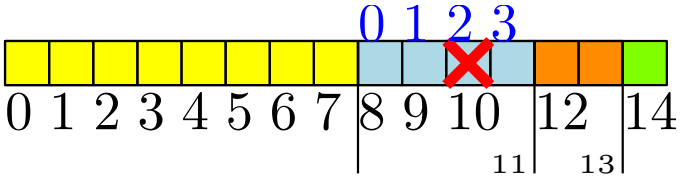


# Proof

1. $m$: A sum of unique powers of $2$, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.
2. Example:



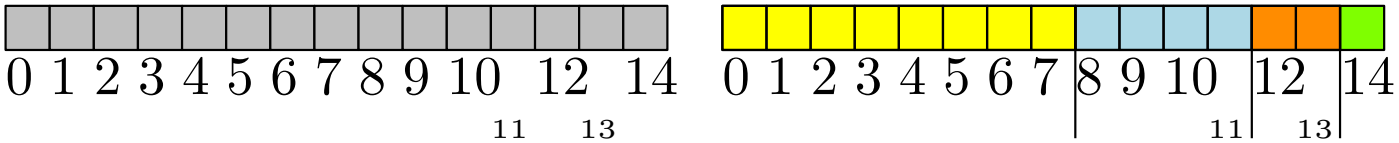

3. decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of $2$.
4. If $x$ is in block $2^k$, output its relative location in the block in binary representation.
5. Example: $x = 10$:



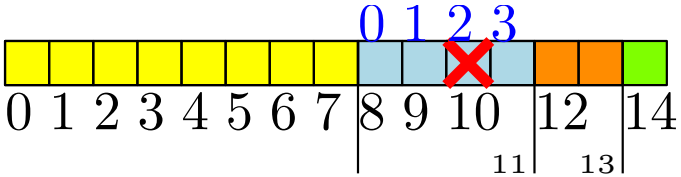

then falls into block $2^2$...
$x$ relative location is 2. Output $2$ written using two bits,
Output: "10".

# Proof

1. $m$: A sum of unique powers of $2$, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.
2. Example:




3. decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of $2$.
4. If $x$ is in block $2^k$, output its relative location in the block in binary representation.
5. Example: $x = 10$:




then falls into block $2^2$...
$x$ relative location is 2. Output $2$ written using two bits,
Output: "10".

# Proof continued

1. Valid extractor...
2. Theorem holds if $m$ is a power of two. Only one block.
3. $m$ not a power of $2$...
4. $X$ falls in block of size $2^k$: then output $k$ complete random bits..
   ... entropy is $k$.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
   There must be a block of size $u$ in the decomposition of $m$.
7. two blocks in decomposition of $m$: sizes $2^k$ and $2^u$.
8. Largest two blocks...
9. $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
10. $Y$: random variable $=$ number of bits output by extractor.

# Proof continued

1. Valid extractor...
2. Theorem holds if $m$ is a power of two. Only one block.
3. $m$ not a power of $2$...
4. $X$ falls in block of size $2^k$: then output $k$ complete random bits..
   ... entropy is $k$.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
   There must be a block of size $u$ in the decomposition of $m$.
7. two blocks in decomposition of $m$: sizes $2^k$ and $2^u$.
8. Largest two blocks...
9. $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
10. $Y$: random variable $=$ number of bits output by extractor.

# Proof continued

1. Valid extractor...
2. Theorem holds if $m$ is a power of two. Only one block.
3. $m$ not a power of $2$...
4. $X$ falls in block of size $2^k$: then output $k$ complete random bits.. ... entropy is $k$.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
   There must be a block of size $u$ in the decomposition of $m$.
7. two blocks in decomposition of $m$: sizes $2^k$ and $2^u$.
8. Largest two blocks...
9. $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
10. $Y$: random variable = number of bits output by extractor.

# Proof continued

1. Valid extractor...
2. Theorem holds if $m$ is a power of two. Only one block.
3. $m$ not a power of $2$...
4. $X$ falls in block of size $2^k$: then output $k$ complete random bits.. ... entropy is $k$.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
   There must be a block of size $u$ in the decomposition of $m$.
7. two blocks in decomposition of $m$: sizes $2^k$ and $2^u$.
8. Largest two blocks...
9. $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
10. $Y$: random variable = number of bits output by extractor.

# Proof continued

1. Valid extractor...
2. Theorem holds if $m$ is a power of two. Only one block.
3. $m$ not a power of $2$...
4. $X$ falls in block of size $2^k$: then output $k$ complete random bits..
   ... entropy is $k$.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
   There must be a block of size $u$ in the decomposition of $m$.
7. two blocks in decomposition of $m$: sizes $2^k$ and $2^u$.
8. Largest two blocks...
9. $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
10. $Y$: random variable $=$ number of bits output by extractor.

# Proof continued

1. Valid extractor...
2. Theorem holds if $m$ is a power of two. Only one block.
3. $m$ not a power of $2$...
4. $X$ falls in block of size $2^k$: then output $k$ complete random bits..
   ... entropy is $k$.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
   There must be a block of size $u$ in the decomposition of $m$.
7. two blocks in decomposition of $m$: sizes $2^k$ and $2^u$.
8. Largest two blocks...
9. $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
10. $Y$: random variable = number of bits output by extractor.

# Proof continued

1. Valid extractor...
2. Theorem holds if $m$ is a power of two. Only one block.
3. $m$ not a power of $2$...
4. $X$ falls in block of size $2^k$: then output $k$ complete random bits..
   ... entropy is $k$.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
   There must be a block of size $u$ in the decomposition of $m$.
7. two blocks in decomposition of $m$: sizes $2^k$ and $2^u$.
8. Largest two blocks...
9. $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
10. $Y$: random variable = number of bits output by extractor.

# Proof continued

1. Valid extractor...
2. Theorem holds if $m$ is a power of two. Only one block.
3. $m$ not a power of $2$...
4. $X$ falls in block of size $2^k$: then output $k$ complete random bits..
   ... entropy is $k$.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
   There must be a block of size $u$ in the decomposition of $m$.
7. two blocks in decomposition of $m$: sizes $2^k$ and $2^u$.
8. Largest two blocks...
9. $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
10. $Y$: random variable = number of bits output by extractor.

# Proof continued

1. Valid extractor...
2. Theorem holds if $m$ is a power of two. Only one block.
3. $m$ not a power of $2$...
4. $X$ falls in block of size $2^k$: then output $k$ complete random bits..
   ... entropy is $k$.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
   There must be a block of size $u$ in the decomposition of $m$.
7. two blocks in decomposition of $m$: sizes $2^k$ and $2^u$.
8. Largest two blocks...
9. $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
10. $Y$: random variable = number of bits output by extractor.

# Proof continued

1. Valid extractor...
2. Theorem holds if $m$ is a power of two. Only one block.
3. $m$ not a power of $2$...
4. $X$ falls in block of size $2^k$: then output $k$ complete random bits..
   ... entropy is $k$.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
   There must be a block of size $u$ in the decomposition of $m$.
7. two blocks in decomposition of $m$: sizes $2^k$ and $2^u$.
8. Largest two blocks...
9. $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
10. $Y$: random variable = number of bits output by extractor.

# Proof continued

1. Valid extractor...
2. Theorem holds if $m$ is a power of two. Only one block.
3. $m$ not a power of $2$...
4. $X$ falls in block of size $2^k$: then output $k$ complete random bits..
   ... entropy is $k$.
5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
   There must be a block of size $u$ in the decomposition of $m$.
7. two blocks in decomposition of $m$: sizes $2^k$ and $2^u$.
8. Largest two blocks...
9. $2^k + 2 * 2^u > m \implies 2^{u+1} + 2^k - m > 0$.
10. $Y$: random variable $=$ number of bits output by extractor.

# Proof continued

1. By lemma, since $\frac{m-2^k}{m} < 1$:

$$\frac{m-2^k}{m} \leq \frac{m-2^k+\left(2^{u+1}+2^k-m\right)}{m+(2^{u+1}+2^k-m)} = \frac{2^{u+1}}{2^{u+1}+2^k}.$$

2. By induction (assumed holds for all numbers smaller than $m$):

$$\mathrm{E}[Y] \geq \frac{2^k}{m}k + \frac{m-2^k}{m}\left(\underbrace{\left\lfloor \lg(m-2^k)\right\rfloor}_{u} - 1\right)$$

$$= \frac{2^k}{m}k + \frac{m-2^k}{m}(\underbrace{k-k}_{=0}+u-1)$$

$$= k + \frac{m-2^k}{m}(u-k-1)$$

# Proof continued

1. By lemma, since $\frac{m-2^k}{m} < 1$:

$$\frac{m-2^k}{m} \leq \frac{m-2^k+\left(2^{u+1}+2^k-m\right)}{m \qquad +(2^{u+1}+2^k-m)} = \frac{2^{u+1}}{2^{u+1}+2^k}.$$

2. By induction (assumed holds for all numbers smaller than $m$):

$$\mathbf{E}[Y] \geq \frac{2^k}{m}k + \frac{m-2^k}{m}\left(\underbrace{\left\lfloor \lg(m-2^k)\right\rfloor}_{u} - 1\right)$$

$$= \frac{2^k}{m}k + \frac{m-2^k}{m}(\underbrace{k-k}_{=0}+u-1)$$

$$= k + \frac{m-2^k}{m}(u-k-1)$$

# Proof continued

1. By lemma, since $\frac{m-2^k}{m} < 1$:

$$\frac{m-2^k}{m} \leq \frac{m-2^k+\left(2^{u+1}+2^k-m\right)}{m \qquad +(2^{u+1}+2^k-m)} = \frac{2^{u+1}}{2^{u+1}+2^k}.$$

2. By induction (assumed holds for all numbers smaller than $m$):

$$\mathbf{E}[Y] \geq \frac{2^k}{m}k + \frac{m-2^k}{m}\left(\underbrace{\left\lfloor \lg(m-2^k)\right\rfloor}_{u} - 1\right)$$

$$= \frac{2^k}{m}k + \frac{m-2^k}{m}(\underbrace{k-k}_{=0} + u - 1)$$

$$= k + \frac{m-2^k}{m}(u-k-1)$$

# Proof continued

1. By lemma, since $\frac{m-2^k}{m} < 1$:

$$\frac{m-2^k}{m} \leq \frac{m-2^k+\left(2^{u+1}+2^k-m\right)}{m \qquad +(2^{u+1}+2^k-m)} = \frac{2^{u+1}}{2^{u+1}+2^k}.$$

2. By induction (assumed holds for all numbers smaller than $m$):

$$\mathbf{E}[Y] \geq \frac{2^k}{m}k + \frac{m-2^k}{m}\left(\underbrace{\left\lfloor \lg(m-2^k)\right\rfloor}_{u} - 1\right)$$

$$= \frac{2^k}{m}k + \frac{m-2^k}{m}(\underbrace{k-k}_{=0} + u - 1)$$

$$= k + \frac{m-2^k}{m}(u-k-1)$$

# Proof continued..

1. We have:

$$\mathbf{E}[Y] \geq k + \frac{m - 2^k}{m}(u - k - 1)$$

$$\geq k + \frac{2^{u+1}}{2^{u+1} + 2^k}(u - k - 1)$$
$$= k - \frac{2^{u+1}}{2^{u+1} + 2^k}(1 + k - u),$$

since $u - k - 1 \leq 0$ as $k > u$.

2. If $u = k - 1$, then $\mathbf{E}[Y] \geq k - \frac{1}{2} \cdot 2 = k - 1$, as required.
3. If $u = k - 2$ then $\mathbf{E}[Y] \geq k - \frac{1}{3} \cdot 3 = k - 1$.

# Proof continued..

1. We have:

$$\mathbf{E}[Y] \geq k + \frac{m - 2^k}{m}(u - k - 1)$$

$$\geq k + \frac{2^{u+1}}{2^{u+1} + 2^k}(u - k - 1)$$
$$= k - \frac{2^{u+1}}{2^{u+1} + 2^k}(1 + k - u),$$

since $u - k - 1 \leq 0$ as $k > u$.

2. If $u = k - 1$, then $\mathbf{E}[Y] \geq k - \frac{1}{2} \cdot 2 = k - 1$, as required.
3. If $u = k - 2$ then $\mathbf{E}[Y] \geq k - \frac{1}{3} \cdot 3 = k - 1$.

# Proof continued..

1. We have:

$$\mathbf{E}[Y] \geq k + \frac{m - 2^k}{m}(u - k - 1)$$

$$\begin{aligned} &\geq k + \frac{2^{u+1}}{2^{u+1} + 2^k}(u - k - 1) \\ &= k - \frac{2^{u+1}}{2^{u+1} + 2^k}(1 + k - u), \end{aligned}$$

   since $u - k - 1 \leq 0$ as $k > u$.

2. If $u = k - 1$, then $\mathbf{E}[Y] \geq k - \frac{1}{2} \cdot 2 = k - 1$, as required.
3. If $u = k - 2$ then $\mathbf{E}[Y] \geq k - \frac{1}{3} \cdot 3 = k - 1$.

# Proof continued..

1. We have:

$$\mathbf{E}[Y] \geq k + \frac{m - 2^k}{m}(u - k - 1)$$

$$\begin{aligned} &\geq k + \frac{2^{u+1}}{2^{u+1} + 2^k}(u - k - 1) \\ &= k - \frac{2^{u+1}}{2^{u+1} + 2^k}(1 + k - u), \end{aligned}$$

   since $u - k - 1 \leq 0$ as $k > u$.
2. If $u = k - 1$, then $\mathbf{E}[Y] \geq k - \frac{1}{2} \cdot 2 = k - 1$, as required.
3. If $u = k - 2$ then $\mathbf{E}[Y] \geq k - \frac{1}{3} \cdot 3 = k - 1$.

# Proof continued.....

1. $\mathbf{E}[Y] \geq k - \frac{2^{u+1}}{2^{u+1}+2^k}(1+k-u)$.
   And $u - k - 1 \leq 0$ as $k > u$.
2. If $u < k - 2$ then

$$\begin{aligned}
\mathbf{E}[Y] &\geq k - \frac{2^{u+1}}{2^k}(1+k-u) \\
&= k - \frac{k-u+1}{2^{k-u-1}} \\
&= k - \frac{2+(k-u-1)}{2^{k-u-1}} \\
&\geq k - 1,
\end{aligned}$$

since $(2+i)/2^i \leq 1$ for $i \geq 2$.

# Proof continued.....

1. $\mathbf{E}[Y] \geq k - \frac{2^{u+1}}{2^{u+1}+2^k}(1+k-u)$.
   And $u - k - 1 \leq 0$ as $k > u$.
2. If $u < k - 2$ then

$$\begin{aligned}
\mathbf{E}[Y] &\geq k - \frac{2^{u+1}}{2^k}(1+k-u) \\
&= k - \frac{k-u+1}{2^{k-u-1}} \\
&= k - \frac{2+(k-u-1)}{2^{k-u-1}} \\
&\geq k - 1,
\end{aligned}$$

since $(2+i)/2^i \leq 1$ for $i \geq 2$.

# Notes

# Notes

# Notes

# Notes