Chapter 24

Approximate Max Cut

CS 573: Algorithms, Fall 2013
November 19, 2013

24.1 Normal distribution

24.1.0.1 Normal distribution – proof

$$
\begin{aligned}
\tau^2 &= \left(\int_{x=-\infty}^{\infty} \exp\left(-\frac{x^2}{2} \right) \mathrm{d}x \right)^2 \\
&= \int_{(x,y)\in \mathbb{R}^2} \exp\left(-\frac{x^2+y^2}{2} \right) \mathrm{d}x \mathrm{d}y \qquad \text{Change of vars: } \begin{array}{c} x = r\cos\alpha, \\ y = r\sin\alpha \end{array} \\
&= \int_{\alpha=0}^{2\pi} \int_{r=0}^{\infty} \exp\left(-\frac{r^2}{2} \right) \left| \det \begin{pmatrix} \frac{\partial r\cos\alpha}{\partial r} & \frac{\partial r\cos\alpha}{\partial \alpha} \\ \frac{\partial r\sin\alpha}{\partial r} & \frac{\partial r\sin\alpha}{\partial \alpha} \end{pmatrix} \right| \mathrm{d}r\, \mathrm{d}\alpha \\
&= \int_{\alpha=0}^{2\pi} \int_{r=0}^{\infty} \exp\left(-\frac{r^2}{2} \right) \left| \det \begin{pmatrix} \cos\alpha & -r\sin\alpha \\ \sin\alpha & r\cos\alpha \end{pmatrix} \right| \mathrm{d}r\, \mathrm{d}\alpha \\
&= \int_{\alpha=0}^{2\pi} \int_{r=0}^{\infty} \exp\left(-\frac{r^2}{2} \right) r\, \mathrm{d}r\, \mathrm{d}\alpha \\
&= \int_{\alpha=0}^{2\pi} \left[-\exp\left(-\frac{r^2}{2} \right) \right]_{r=0}^{\infty} \mathrm{d}\alpha = \int_{\alpha=0}^{2\pi} 1\, \mathrm{d}\alpha = 2\pi
\end{aligned}
$$

24.1.0.2 Multidimensional normal distribution

(A) A random variable X has ***normal distribution*** if $\mathbf{Pr}[X = x] = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$.
(B) $X \sim N(0,1)$.
(C) A vector $\mathbf{x} = (x_1, \ldots, x_n)$ has d-dimensional normal distributed (i.e., $\mathbf{v} \sim N^n(0,1)$ if $v_1, \ldots, v_n \tilde{N}(0,1)$
(D) Consider a vector $\mathbf{v} \in \mathbb{R}^n$, such that $\|\mathbf{v}\| = 1$. Let $\mathbf{x} \sim N^n(0,1)$. Then $z = \langle \mathbf{v}, \mathbf{x} \rangle$ has normal distribution!

24.2 Approximate Max Cut

24.2.1 The movie so far...

24.2.1.1 Summary: It sucks.

(A) Seen: Examples of using rounding techniques for approximation.
(B) So far: Relaxed optimization problem is LP.
(C) But... We know how to solve ***convex programming***.
(D) Convex programming $\gg$ LP.
(E) Convex programming can be solved in polynomial time.
(F) Solving convex programming is outside scope: assume doable in polynomial time.
(G) Today's lecture:
  (A) Revisit **MAX CUT**.
  (B) Show how to relax it into semi-definite programming problem.
  (C) Solve relaxation.
  (D) Show how to round the relaxed problem.

24.2.2 Problem Statement: MAX CUT

24.2.2.1 Since this is a theory class, we will define our problem.

(A) $\mathsf{G} = (\mathsf{V}, \mathsf{E})$: undirected graph.
(B) $\forall ij \in \mathsf{E}$: nonnegative weights ω_{ij}.
(C) **MAX CUT** (***maximum cut problem***): Compute set $S \subseteq \mathsf{V}$ maximizing weight of edges in cut $\left(S, \overline{S}\right)$.
(D) $ij \notin \mathsf{E} \implies \omega_{ij} = O$.
(E) ***weight*** of cut: $w\left(S, \overline{S}\right) = \sum_{i \in S,\ j \in \overline{S}} \omega_{ij}$.
(F) Known: problem is **NP-Complete**.
Hard to approximate within a certain constant.

24.2.3 Max cut as integer program

24.2.3.1 because what can go wrong?

(A) Vertices: $\mathsf{V} = \{1, \ldots, n\}$.
(B) ω_{ij}: non-negative weights on edges.
(C) max cut $w(S, \overline{S})$ is computed by the integer quadratic program:

$$
\begin{array}{llll}
(\mathrm{Q}) & \max & \dfrac{1}{2} \sum_{i<j} \omega_{ij}(1 - y_i y_j) & \\
 & \text{subject to:} & y_i \in \{-1, 1\} & \forall i \in \mathsf{V}.
\end{array}
$$

(D) Set: $S = \left\{ i \;\middle|\; y_i = 1 \right\}$.
(E) $\omega\left(S, \overline{S}\right) = \frac{1}{2} \sum_{i<j} \omega_{ij}(1 - y_i y_j)$.

24.2.4 Relaxing $-1, 1$...

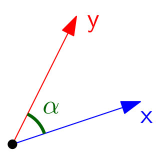
24.2.4.1 Because 1 and -1 are just vectors.

(A) Solving quadratic integer programming is of course **NP-Hard**.
(B) Want a relaxation...
(C) 1 and -1 are just roots of unity.
(D) FFT: All roots of unity are a circle.
(E) In higher dimensions: All unit vectors are points on unit sphere.
(F) y_i are just unit vectors.
(G) $y_i * y_j$ is replaced by dot product $\langle y_i, y_j \rangle$.

24.2.5 Quick reminder about dot products

24.2.5.1 Because not everybody remembers what they did in kindergarten

(A) $\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{y} = (y_1, \ldots, y_d)$.
(B) $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{d} x_i y_i$.
(C) For a vector $\mathbf{v} \in \mathbb{R}^d$: $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle$.
(D) $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos \alpha$.
α: Angle between $\mathbf{x}$ and $\mathbf{y}$.



(E) $\mathbf{x} \perp \mathbf{y}$: $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
(F) $\mathbf{x} = \mathbf{y}$ and $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$: $\langle \mathbf{x}, \mathbf{y} \rangle = 1$.
(G) $\mathbf{x} = -\mathbf{y}$ and $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$: $\langle \mathbf{x}, \mathbf{y} \rangle = -1$.

24.2.6 Relaxing $-1, 1$...

24.2.6.1 Because 1 and -1 are just vectors.

(A) max cut $w(S, \overline{S})$ as integer quadratic program:

$$
\begin{aligned}
(Q) \qquad & \max \quad \frac{1}{2} \sum_{i<j} \omega_{ij} (1 - y_i y_j) \\
& \text{subject to:} \quad y_i \in \{-1, 1\} \qquad \forall i \in \mathsf{V}.
\end{aligned}
$$

(B) Relaxed semi-definite programming version:

$$
\begin{aligned}
(P) \qquad & \max \quad \gamma = \frac{1}{2} \sum_{i<j} \omega_{ij} \left(1 - \langle v_i, v_j \rangle\right) \\
& \text{subject to:} \quad v_i \in \mathbb{S}^{(n)} \qquad \forall i \in V,
\end{aligned}
$$

$\mathbb{S}^{(n)}$: n dimensional unit sphere in $\mathbb{R}^{n+1}$.

24.2.6.2 Discussion...

(A) semi-definite programming: special case of convex programming.

(B) Can be solved in polynomial time.
(C) Solve within a factor of $(1+\varepsilon)$ of optimal, for any $\varepsilon > 0$, in polynomial time.
(D) Intuition: vectors of one side of the cut, and vertices on the other sides, would have faraway vectors.

24.2.6.3 Approximation algorithm

(A) Given instance, compute SDP (P).
(B) Compute optimal solution for (P).
(C) generate a random vector $\vec{r}$ on the unit sphere $\mathbb{S}^{(n)}$.
(D) induces hyperplane $h \qquad \equiv \qquad \langle \vec{r}, \mathbf{x} \rangle = 0$
(E) assign all vectors on one side of h to S, and rest to $\overline{S}$.

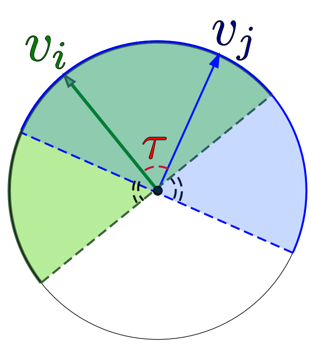
$$S = \left\{ v_i \;\middle|\; \langle v_i, \vec{r} \rangle \geq 0 \right\}.$$

24.2.7 Analysis

24.2.7.1 Analysis...

Intuition: with good probability, vectors in the solution of (P) that have large angle between them would be separated by cut.

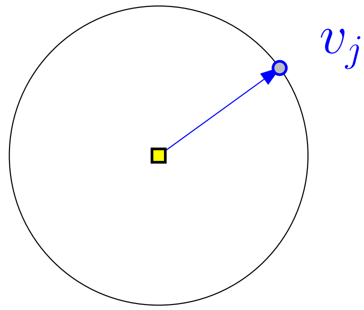
Lemma 24.2.1. $\mathbf{Pr}\Big[\mathrm{sign}\Big(\langle v_i, \vec{r}\,\rangle\Big) \neq \mathrm{sign}(\langle v_j, \vec{r}\,\rangle)\Big] = \dfrac{1}{\pi} \arccos\Big(\langle v_i, v_j \rangle\Big) = \dfrac{\tau}{\pi}.$



24.2.7.2 Proof...

(A) Think v_i, v_j and $\vec{r}$ as being in the plane.
(B) ... reasonable assumption!
  (A) g: plane spanned by v_i and v_j.
  (B) Only care about signs of $\langle v_i, \vec{r} \rangle$ and $\langle v_j, \vec{r} \rangle$
  (C) can be decided by projecting $\vec{r}$ on g... and normalizing it to have length 1.
  (D) Sphere is symmetric $\implies$ sampling $\vec{r}$ from $\mathbb{S}^{(n)}$ projecting it down to g, and then normalizing it
  $\equiv$ choosing uniformly a vector from the unit circle in g

24.2.7.3 Proof via figure...



$\tau = \arccos\Big(\langle v_i, v_j\rangle\Big)$

24.2.7.4 Proof...

(A) Think v_i, v_j and $\vec{r}$ as being in the plane.
(B) $\mathrm{sign}(\langle v_i, \vec{r}\,\rangle) \neq \mathrm{sign}(\langle v_j, \vec{r}\,\rangle)$ happens only if $\vec{r}$ falls in the double wedge formed by the lines perpendicular to v_i and v_j.
(C) angle of double wedge = angle τ between v_i and v_j.
(D) v_i and v_j are unit vectors: $\langle v_i, v_j\rangle = \cos(\tau)$.
$\tau = \angle v_i v_j$.
(E) Thus,

$$\begin{aligned}\mathbf{Pr}\Big[\mathrm{sign}(\langle v_i, \vec{r}\,\rangle) \neq \mathrm{sign}(\langle v_j, \vec{r}\,\rangle)\Big] &= \frac{2\tau}{2\pi} \\ &= \frac{1}{\pi} \cdot \arccos(\langle v_i, v_j\rangle)\,,\end{aligned}$$

as claimed. ■

24.2.7.5 Theorem

Theorem 24.2.2. *Let W be the random variable which is the weight of the cut generated by the algorithm. We have*

$$\mathbf{E}[W] = \frac{1}{\pi}\sum_{i<j}\omega_{ij}\arccos(\langle v_i, v_j\rangle)\,.$$

24.2.7.6 Proof

(A) X_{ij}: indicator variable $= 1 \iff$ edge ij is in the cut.
(B) $\mathbf{E}[X_{ij}] = \mathbf{Pr}\Big[\mathrm{sign}(\langle v_i, \vec{r}\,\rangle) \neq \mathrm{sign}(\langle v_j, \vec{r}\,\rangle)\Big]$
$= \frac{1}{\pi}\arccos\Big(\langle v_i, v_j\rangle\Big)$, by lemma.
(C) $W = \sum_{i<j}\omega_{ij}X_{ij}$, and by linearity of expectation...

$$\mathbf{E}[W] = \sum_{i<j}\omega_{ij}\,\mathbf{E}[X_{ij}] = \frac{1}{\pi}\sum_{i<j}\omega_{ij}\arccos\Big(\langle v_i, v_j\rangle\Big)\,.$$

■

24.2.7.7 Lemma

Lemma 24.2.3. *For* $-1 \le y \le 1$*, we have* $\dfrac{\arccos(y)}{\pi} \ge \alpha \cdot \dfrac{1}{2}(1-y)$*, where* $\alpha = \min_{0 \le \psi \le \pi} \dfrac{2}{\pi} \dfrac{\psi}{1-\cos(\psi)}$.

Proof: Set $y = \cos(\psi)$. The inequality now becomes $\frac{\psi}{\pi} \ge \alpha \frac{1}{2}(1-\cos\psi)$. Reorganizing, the inequality becomes $\frac{2}{\pi}\frac{\psi}{1-\cos\psi} \ge \alpha$, which trivially holds by the definition of α. ■

24.2.7.8 Lemma

Lemma 24.2.4. $\alpha > 0.87856$.

Proof: Using simple calculus, one can see that α achieves its value for $\psi = 2.331122...$, the nonzero root of $\cos\psi + \psi \sin\psi = 1$. ■

24.2.7.9 Result

Theorem 24.2.5. *The above algorithm computes in expectation a cut with total weight* $\alpha \cdot \mathrm{Opt} \ge 0.87856\mathrm{Opt}$*, where* Opt *is the weight of the maximal cut.*

Proof: Consider the optimal solution to (P), and lets its value be $\gamma \ge \mathrm{Opt}$. By lemma:

$$\begin{aligned}\mathbf{E}[W] &= \frac{1}{\pi}\sum_{i<j} \omega_{ij} \arccos(\langle v_i, v_j \rangle) \\ &\ge \sum_{i<j} \omega_{ij} \alpha \frac{1}{2}(1 - \langle v_i, v_j \rangle) = \alpha\gamma \ge \alpha \cdot \mathrm{Opt}. \qquad \blacksquare\end{aligned}$$

■

24.3 Semi-definite programming

24.3.0.10 SDP: Semi-definite programming

(A) $x_{ij} = \langle v_i, v_j \rangle$.
(B) M: $n \times n$ matrix with x_{ij} as entries.
(C) $x_{ii} = 1$, for $i = 1, \ldots, n$.
(D) V: matrix having vectors $v_1, \ldots, v_n$ as its columns.
(E) $M = V^T V$.
(F) $\forall v \in \mathbb{R}^n$: $v^T M v = v^T A^T A v = (Av)^T(Av) \ge 0$.
(G) M is ***positive semidefinite*** (PSD).
(H) Fact: Any PSD matrix P can be written as $P = B^T B$.
(I) Furthermore, given such a matrix P of size $n \times n$, we can compute B such that $P = B^T B$ in $O(n^3)$ time.
(J) Known as ***Cholesky decomposition***.

24.3.0.11 SDP: Semi-definite programming

(A) If PSD $P = B^T B$ has a diagonal of 1
(B) $\implies$ B has columns which are unit vectors.
(C) If solve SDP (P), get back semi-definite matrix...
(D) ... recover the vectors realizing the solution (i.e., compute B)
(E) Now, do the rounding.
(F) SDP (P) can be restated as

$$
\begin{aligned}
(SD) \qquad\qquad & \max & & \frac{1}{2}\sum_{i<j} \omega_{ij}(1 - x_{ij}) \\
& \text{subject to:} & & x_{ii} = 1 \qquad \text{for } i = 1, \ldots, n \\
& & & \Big(x_{ij}\Big)_{i=1,\ldots,n, j=1,\ldots,n} \quad \text{is a PSD matrix.}
\end{aligned}
$$

24.3.0.12 SDP: Semi-definite programming

(A) SDP is

$$
\begin{aligned}
(SD) \qquad\qquad & \max & & \frac{1}{2}\sum_{i<j} \omega_{ij}(1 - x_{ij}) \\
& \text{subject to:} & & x_{ii} = 1 \qquad \text{for } i = 1, \ldots, n \\
& & & \Big(x_{ij}\Big)_{i=1,\ldots,n, j=1,\ldots,n} \quad \text{is a PSD matrix.}
\end{aligned}
$$

(B) find optimal value of a linear function...
(C) ... over a set which is the intersection of:
  (A) linear constraints, and
  (B) set of positive semi-definite matrices.

24.3.0.13 Lemma

Lemma 24.3.1. *Let $\mathcal{U}$ be the set of $n \times n$ positive semidefinite matrices. The set $\mathcal{U}$ is convex.*

Proof: Consider $A, B \in \mathcal{U}$, and observe that for any $t \in [0, 1]$, and vector $v \in \mathbb{R}^n$, we have:

$$
\begin{aligned}
v^T \Big(tA + (1-t)B\Big)\, v &= v^T \Big(tAv + (1-t)Bv\Big) \\
&= tv^T Av + (1-t)v^T Bv \geq 0 + 0 \geq 0,
\end{aligned}
$$

since A and B are positive semidefinite. ∎

24.3.0.14 More on positive semidefinite matrices

(A) PSD matrices corresponds to ellipsoids.
(B) $x^T Ax = 1$: the set of vectors solve this equation is an ellipsoid.
(C) Eigenvalues of a PSD are all non-negative real numbers.
(D) Given matrix: can in polynomial time decide if it is PSD.
(E) ... by computing the eigenvalues of the matrix.
(F) $\implies$ SDP: optimize a linear function over a convex domain.
(G) SDP can be solved using interior point method, or the ellipsoid method.
(H) See Boyd and Vandenberghe [2004], Grötschel et al. [1993] for more details.
(I) Membership oracle: ability to decide in polynomial time, given a solution, whether its feasible or not.

24.4 Bibliographical Notes

24.4.0.15 Bibliographical Notes

(A) Approx. algorithm presented by Goemans and Williamson Goemans and Williamson [1995].
(B) Håstad Håstad [2001] showed that MAX CUT can not be approximated within a factor of $16/17 \approx 0.941176$.
(C) Khot etal Khot et al. [2004] showed a hardness result that matches the constant of Goemans and Williamson (i.e., one can not approximate it better than α, unless $\mathbf{P} = \mathbf{NP}$).

24.4.0.16 Bibliographical Notes

(A) Relies on two conjectures: "Unique Games Conjecture" and "Majority is Stablest".
(B) "Majority is Stablest" conjecture was proved by Mossel etal Mossel et al. [2005].
(C) Not clear if the "Unique Games Conjecture" is true, see the discussion in Khot et al. [2004].
(D) Goemans and Williamson work spurred wide research on using SDP for approximation algorithms.

Bibliography

S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge, 2004. URL `http://www.stanford.edu/~boyd/cvxbook/`.

M. X. Goemans and D. P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. Assoc. Comput. Mach.*, 42(6):1115–1145, November 1995.

M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*, volume 2 of *Algorithms and Combinatorics.* Springer-Verlag, Berlin Heidelberg, 2nd edition, 1993.

J. Håstad. Some optimal inapproximability results. *J. Assoc. Comput. Mach.*, 48(4):798–859, 2001. ISSN 0004-5411. doi: http://doi.acm.org/10.1145/502090.502098.

S. Khot, G. Kindler, E. Mossel, and R. O'Donnell. Optimal inapproximability results for max cut and other 2-variable csps. In *Proc. 45th Annu. IEEE Sympos. Found. Comput. Sci.* (FOCS), pages 146–154, 2004. To appear in SICOMP.

E. Mossel, R. O'Donnell, and K. Oleszkiewicz. Noise stability of functions with low influences invariance and optimality. In *Proc. 46th Annu. IEEE Sympos. Found. Comput. Sci.* (FOCS), pages 21–30, 2005.