Approximation Algorithms using Linear Programming

Lecture 20

November 5, 2013

Part I

Weighted vertex cover

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Weighted Vertex Cover problem

$\mathbf{G} = (\mathbf{V}, \mathbf{E})$.
Each vertex $\mathbf{v} \in \mathbf{V}$: cost $\mathbf{c_v}$.
Compute a vertex cover of minimum cost.

1. vertex cover: subset of vertices $\mathbf{V}$ so each edge is covered.
2. NP-Hard
3. ...unweighted Vertex Cover problem.
4. ... write as an integer program (IP):
5. $\forall \mathbf{v} \in \mathbf{V}$: $x_{\mathbf{v}} = 1 \iff \mathbf{v}$ in the vertex cover.
6. $\forall \mathbf{vu} \in \mathbf{E}$: covered. $\implies x_{\mathbf{v}} \vee x_{\mathbf{u}}$ **true**. $\implies x_{\mathbf{v}} + x_{\mathbf{u}} \geq 1$.
7. minimize total cost: $\min \sum_{\mathbf{v} \in \mathbf{V}} x_{\mathbf{v}} \mathbf{c_v}$.

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$$\begin{aligned} \min \quad & \sum_{v \in V} c_v x_v, & & \\ \text{such that} \quad & x_v \in \{0,1\} & \forall v \in V & \qquad (1) \\ & x_v + x_u \geq 1 & \forall vu \in E. & \end{aligned}$$

1. ... **NP-Hard**.
2. relax the integer program.
3. allow x_v get values $\in [0,1]$.
4. $x_v \in \{0,1\}$ replaced by $0 \leq x_v \leq 1$. The resulting LP is

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Weighted vertex cover – rounding the LP

1. Optimal solution to this LP: $\widehat{x_v}$ value of var X_v, $\forall v \in V$.
2. optimal value of LP solution is $\hat{\alpha} = \sum_{v \in V} c_v \widehat{x_v}$.
3. optimal integer solution: x_v^I, $\forall v \in V$ and α^I.
4. **Any valid solution to IP is valid solution for LP!**
5. $\hat{\alpha} \leq \alpha^I$.

   Integral solution not better than LP.
6. Got fractional solution (i.e., values of $\widehat{x_v}$).
7. Fractional solution is better than the optimal cost.
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How to round?

1. consider vertex **v** and fractional value $\widehat{x_{\mathsf{v}}}$.
2. If $\widehat{x_{\mathsf{v}}} = 1$ then include in solution!
3. If $\widehat{x_{\mathsf{v}}} = 0$ then do include in solution.
4. if $\widehat{x_{\mathsf{v}}} = 0.9 \implies \mathrm{LP}$ considers **v** as being 0.9 useful.
5. The LP puts its money where its belief is...
6. ...$\widehat{\alpha}$ value is a function of this "belief" generated by the LP.
7. Big idea: Trust LP values as guidance to usefulness of vertices.
8. Pick all vertices $\geq$ threshold of usefulness according to LP.
9. $S = \left\{ \mathsf{v} \;\middle|\; \widehat{x_{\mathsf{v}}} \geq 1/2 \right\}$.
10. **Claim**: S a valid vertex cover, and cost is low.
11. Indeed, edge cover as: $\forall \mathsf{vu} \in \mathsf{E}$ have $\widehat{x_{\mathsf{v}}} + \widehat{x_{\mathsf{u}}} \geq 1$.
12. $\widehat{x_{\mathsf{v}}}, \widehat{x_{\mathsf{u}}} \in (0, 1)$

    $\implies \widehat{x_{\mathsf{v}}} \geq 1/2$ or $\widehat{x_{\mathsf{u}}} \geq 1/2$.

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    $\implies S$ covers all the edges of G.

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α^I is cost of the optimal solution $\implies$

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*The **Weighted Vertex Cover** problem can be 2-approximated by solving a single LP. Assuming computing the LP takes polynomial time, the resulting approximation algorithm takes polynomial time.*

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The lessons we can take away

Or not - boring, boring, boring.

1. Weighted vertex cover is simple, but resulting approximation algorithm is non-trivial.
2. Not aware of any other 2-approximation algorithm does not use LP. (For the weighted case!)
3. Solving a ***relaxation*** of an optimization problem into a LP provides us with insight.
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Part II

Revisiting Set Cover

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1. Purpose: See new technique for an approximation algorithm.
2. Not better than greedy algorithm already seen $O(\log n)$ approximation.

Set Cover

Instance: $(S, \mathcal{F})$

S - a set of n elements

$\mathcal{F}$ - a family of subsets of S, s.t. $\bigcup_{X \in \mathcal{F}} X = S$.

Question: The set $\mathcal{X} \subseteq F$ such that $\mathcal{X}$ contains as few sets as possible, and $\mathcal{X}$ covers S.

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$$\begin{aligned} \min \quad & \alpha = \sum_{U \in \mathcal{F}} x_U, & \\ \text{s.t.} \quad & x_U \in \{0,1\} & \forall U \in \mathcal{F}, \\ & \sum_{U \in \mathcal{F}, s \in U} x_U \geq 1 & \forall s \in S. \end{aligned}$$

Next, we relax this IP into the following LP.

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3. Use LP solution to guide in rounding process.
4. If $\widehat{x_U}$ is close to 1 then pick U to cover.
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6. Idea: Pick $U \in \mathcal{F}$ by randomly choosing it with ***probability*** $\widehat{x_U}$.
7. Resulting family of sets $\mathcal{G}$.
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1. **Solution**: Repeat rounding stage $m = 10 \lceil \lg n \rceil = O(\log n)$ times.
2. $n = |S|$.
3. $\mathcal{G}_i$: random cover computed in ith iteration.
4. $\mathcal{H} = \cup_i \mathcal{G}_i$. Return $\mathcal{H}$ as the required cover.

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1. For an element $s \in S$, we have that
$$\sum_{U \in \mathcal{F}, s \in U} \widehat{x_U} \geq 1, \tag{2}$$
2. probability s not covered by $\mathcal{G}_i$ (ith iteration set).
$\mathbf{Pr}\Big[s \text{ not covered by } \mathcal{G}_i \Big]$
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$= \exp\Big(- \sum_{U \in \mathcal{F}, s \in U} \widehat{x_U} \Big) \leq \exp(-1) \leq \frac{1}{2},$
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2. $\implies$ Each iteration expected cost of cover $\leq$ cost of optimal solution (i.e., α^I).
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The result

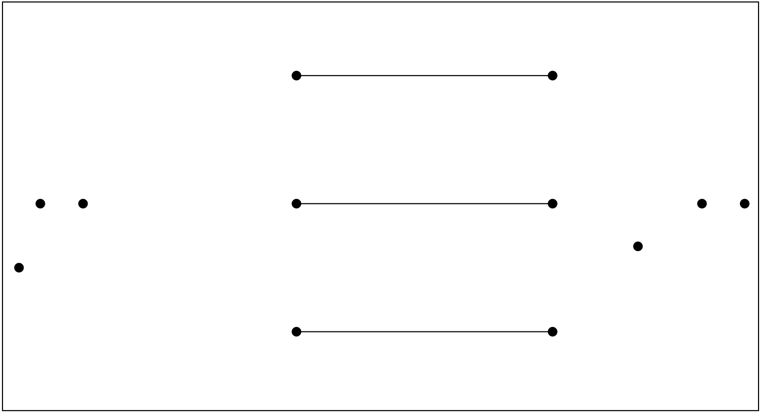
Theorem

By solving an LP *one can get an* $O(\log n)$*-approximation to set cover by a randomized algorithm. The algorithm succeeds with high probability.*

Part III

Minimizing congestion

Minimizing congestion by example



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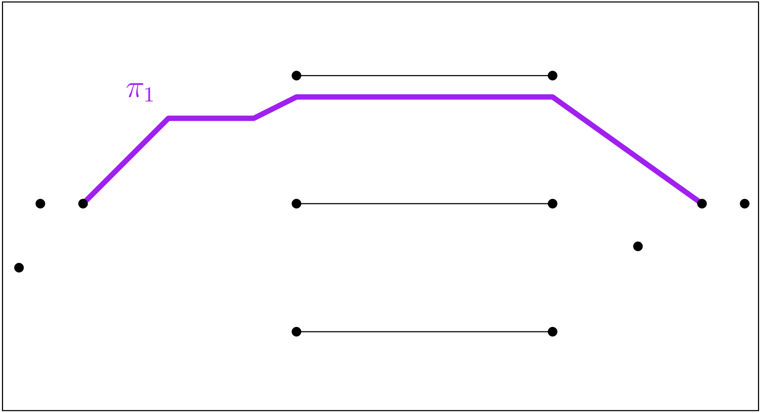

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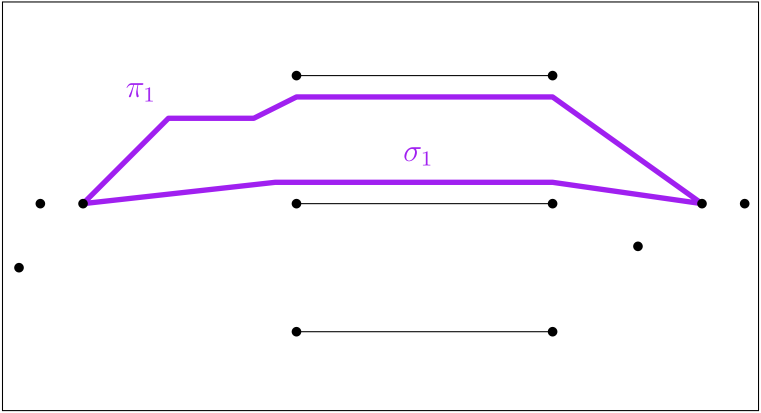

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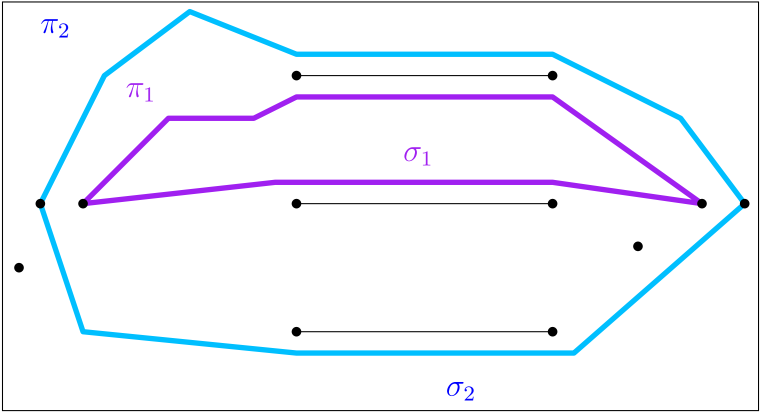

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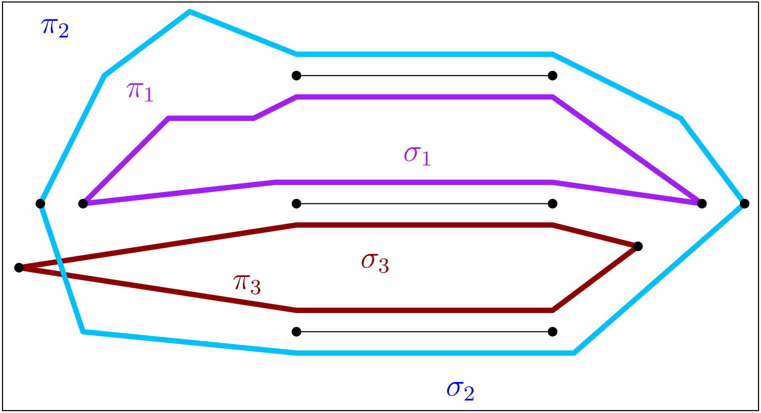

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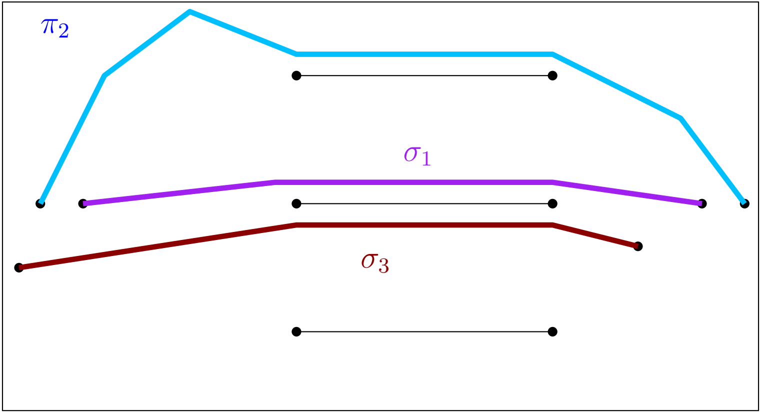

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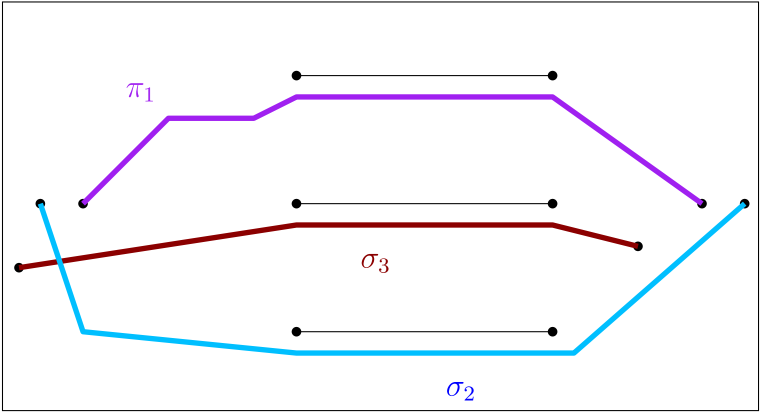

Minimizing congestion

1. $\mathbf{G}$: graph. n vertices.
2. π_i, σ_i paths with the same endpoints $\mathbf{v}_i, \mathbf{u}_i \in \mathbf{V}(\mathbf{G})$, for $i = 1, \ldots, t$.
3. Rule I: Send one unit of flow from $\mathbf{v}_i$ to $\mathbf{u}_i$.
4. Rule II: Choose whether to use π_i or σ_i.
5. Target: No edge in $\mathbf{G}$ is being used too much.

Definition

Given a set X of paths in a graph $\mathbf{G}$, the ***congestion*** of X is the maximum number of paths in X that use the same edge.

Minimizing congestion

1. IP $\implies$ LP:

$$\begin{aligned} \min \quad & w & \\ \text{s.t.} \quad & x_i \geq 0 & i = 1, \ldots, t, \\ & x_i \leq 1 & i = 1, \ldots, t, \\ & \sum_{\mathbf{e} \in \pi_i} x_i + \sum_{\mathbf{e} \in \sigma_i} (1 - x_i) \leq w & \forall \mathbf{e} \in E. \end{aligned}$$

2. $\widehat{x_i}$: value of x_i in the optimal LP solution.
3. $\widehat{w}$: value of w in LP solution.
4. Optimal congestion must be bigger than $\widehat{w}$.
5. X_i: random variable one with probability $\widehat{x_i}$, and zero otherwise.
6. If $X_i = 1$ then use π to route from $\mathbf{v}_i$ to $\mathbf{u}_i$.
7. Otherwise use σ_i.

Minimizing congestion

1. Congestion of **e** is

$$Y_{\mathsf{e}} = \sum_{\mathsf{e}\in\pi_i} X_i + \sum_{\mathsf{e}\in\sigma_i} (1 - X_i).$$

2. And in expectation

$$\begin{aligned}\alpha_{\mathsf{e}} = \mathbf{E}\Big[Y_{\mathsf{e}}\Big] &= \mathbf{E}\left[\sum_{\mathsf{e}\in\pi_i} X_i + \sum_{\mathsf{e}\in\sigma_i} (1 - X_i)\right] \\ &= \sum_{\mathsf{e}\in\pi_i} \mathbf{E}\Big[X_i\Big] + \sum_{\mathsf{e}\in\sigma_i} \mathbf{E}\Big[(1 - X_i)\Big] \\ &= \sum_{\mathsf{e}\in\pi_i} \widehat{x_i} + \sum_{\mathsf{e}\in\sigma_i} (1 - \widehat{x_i}) \leq \widehat{w}.\end{aligned}$$

Minimizing congestion - continued

1. $Y_e = \sum_{e \in \pi_i} X_i + \sum_{e \in \sigma_i} (1 - X_i)$.
2. Y_e is just a sum of independent $0/1$ random variables!
3. Chernoff inequality tells us sum can not be too far from expectation!

Minimizing congestion - continued

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Minimizing congestion - continued

1. By Chernoff inequality:

$$\Pr\Big[Y_e \geq (1+\delta)\alpha_e \Big] \leq \exp\left(-\frac{\alpha_e \delta^2}{4}\right) \leq \exp\left(-\frac{\widehat{w}\delta^2}{4}\right).$$

2. Let $\delta = \sqrt{\frac{400}{\widehat{w}} \ln t}$. We have that

$$\Pr\Big[Y_e \geq (1+\delta)\alpha_e \Big] \leq \exp\left(-\frac{\delta^2 \widehat{w}}{4}\right) \leq \frac{1}{t^{100}},$$

3. If $t \geq n^{1/50}$ then all the edges in the graph do not have congestion larger than $(1+\delta)\widehat{w}$.

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2. Play with the numbers. If $t = n$, and $\widehat{w} \geq \sqrt{n}$. Then, the solution has congestion larger than the optimal solution by a factor of

$$1 + \delta = 1 + \sqrt{\frac{20}{\widehat{w}} \ln t} \leq 1 + \frac{\sqrt{20 \ln n}}{n^{1/4}},$$

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Minimizing congestion: result

Theorem

Given a graph with n vertices, and t pairs of vertices, such that for every pair (s_i, t_i) there are two possible paths to connect s_i to t_i. Then one can choose for each pair which path to use, such that the most congested edge, would have at most $(1+\delta)\mathrm{opt}$, where opt is the congestion of the optimal solution, and $\delta = \sqrt{\frac{20}{\widehat{w}} \ln t}$.

When the congestion is low

1. Assume $\widehat{w}$ is a constant.
2. Can get a better bound by using the Chernoff inequality in its more general form.
3. set $\delta = c \ln t / \ln \ln t$, where c is a constant. For $\mu = \alpha_e$, we have that

$$\Pr\Big[Y_e \geq (1+\delta)\mu\Big] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$$
$$= \exp\Big(\mu\big(\delta - (1+\delta)\ln(1+\delta)\big)\Big)$$
$$= \exp\Big(-\mu c' \ln t\Big) \leq \frac{1}{t^{O(1)}},$$

where c' is a constant that depends on c and grows if c grows.

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Part IV

Reminder about Chernoff inequality

Chernoff inequality

Problem

Let $X_1, \ldots X_n$ *be* n *independent Bernoulli trials, where*

$$\Pr\left[X_i = 1\right] = p_i, \qquad \Pr\left[X_i = 0\right] = 1 - p_i,$$
$$Y = \sum_i X_i, \qquad \text{and} \qquad \mu = \mathbf{E}\left[Y\right].$$

We are interested in bounding the probability that $Y \geq (1 + \delta)\mu$.

Chernoff inequality

Theorem (Chernoff inequality)

For any $\delta > 0$,

$$\Pr\left[Y > (1+\delta)\mu\right] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$

Or in a more simplified form, for any $\delta \leq 2e - 1$,

$$\Pr\left[Y > (1+\delta)\mu\right] < \exp\left(-\mu\delta^2/4\right),$$

and

$$\Pr\left[Y > (1+\delta)\mu\right] < 2^{-\mu(1+\delta)},$$

for $\delta \geq 2e - 1$.

More Chernoff...

Theorem

Under the same assumptions as the theorem above, we have

$$\Pr\Big[Y < (1-\delta)\mu\Big] \leq \exp\left(-\mu\frac{\delta^2}{2}\right).$$

Notes

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