Linear Programming

Lecture 18

October 29, 2013

Part I

Linear Programming

Economic planning

Guns/nuclear-bombs/napkins/star-wars/professors/butter/mice problem

1. Penguina: a country.
2. Ruler need to decide how to allocate resources.
3. Maximize benefit.
4. Budget allocation
   (i) Nuclear bomb has a tremendous positive effect on security while being expensive.
   (ii) Guns, on the other hand, have a weaker effect.
5. Penguina need to prove a certain level of security:
   $$x_{gun} + 1000 * x_{nuclear-bomb} \geq 1000,$$
   where x_{guns}: # guns $x_{nuclear-bomb}$: # nuclear-bombs constructed.
6. $100 * x_{gun} + 1000000 * x_{nuclear-bomb} \leq x_{security}$
   $x_{security}$: total amount spent on security.
   $100/1,000,000$: price of producing a single gun/nuclear bomb.

Linear programming

An instance of ***linear programming*** (LP):

1. $x_1, \ldots, x_n$: variables.
2. For $j = 1, \ldots, m$: $a_{j1}x_1 + \ldots + a_{jn}x_n \leq b_j$: linear inequality.
3. i.e., ***constraint***.
4. Q: $\exists$ s an assignment of values to $x_1, \ldots, x_n$ such that all inequalities are satisfied.
5. Many possible solutions... Want solution that maximizes some linear quantity.
6. ***objective function***: linear inequality being maximized.

Linear programming – example

$$\boxed{\begin{array}{l} a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n \leq b_2 \\ \ldots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m \\ \max c_1x_1 + \ldots + c_nx_n. \end{array}}$$

History

1. 1939: L. V. Kantorovich noticed the importance of certain type of Linear Programming problems for resource allocation.
2. 1947: Dantzig invented the simplex method for solving LP problems for the US Air force planning problems.
3. 1947: T. C. Koopmans showed LP provide the right model for the analysis of classical economic theories.
4. 1975: Koopmans and Kantorovich got the Nobel prize of economics.
5. Kantorovich the only the Russian economist that got the Nobel prize

Network flow via linear programming

Input: $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with source $\mathbf{s}$ and sink $\mathbf{t}$, and capacities $\mathbf{c}(\cdot)$ on the edges. Compute max flow in $\mathbf{G}$.

$$
\begin{array}{ll}
\forall\,(u \to v) \in E & 0 \leq x_{u \to v} \\
 & x_{u \to v} \leq \mathbf{c}(u \to v) \\
 & \\
\forall v \in V \setminus \{\mathbf{s}, \mathbf{t}\} & \displaystyle\sum_{(u \to v) \in E} x_{u \to v} - \sum_{(v \to w) \in E} x_{v \to w} \leq 0 \\
 & \displaystyle\sum_{(u \to v) \in E} x_{u \to v} - \sum_{(v \to w) \in E} x_{v \to w} \geq 0 \\
 & \\
\text{maximizing} & \sum_{(\mathbf{s} \to u) \in E} x_{\mathbf{s} \to u}
\end{array}
$$

Part II

The Simplex Algorithm

Rewriting an LP

$$\max \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

1. Rewrite: so every variable is non-negative.
2. Replace variable x_i by x_i' and x_i'', where new constraints are: $x_i = x_i' - x_i''$, $\quad x_i' \geq 0$ and $x_i'' \geq 0$.
3. Example: The (silly) LP $2x + y \geq 5$ rewritten:
$2x' - 2x'' + y' - y'' \geq 5$,
$x' \geq 0$, $y' \geq 0$,
$x'' \geq 0$, and
$y'' \geq 0$.

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Rewriting an LP into standard form

Lemma

Given an instance I of LP, one can rewrite it into an equivalent LP, such that all the variables must be non-negative. This takes linear time in the size of I.

An LP where all variables must be non-negative is in ***standard form***

Rewriting an LP into standard form

Lemma

Given an instance $\boldsymbol{I}$ of LP*, one can rewrite it into an equivalent* LP*, such that all the variables must be non-negative. This takes linear time in the size of $\boldsymbol{I}$.*

An LP where all variables must be non-negative is in ***standard form***

Standard form of LP

A linear program in standard form.

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2, \ldots, m \\ & x_j \ge 0 \qquad\qquad\quad \text{for } j = 1, \ldots, n. \end{aligned}$$

Standard form of LP

Because everything is clearer when you use matrices. Not.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(n-1)} & a_{2n} \\ \vdots & \dots & \dots & \dots & \vdots \\ a_{(m-1)1} & a_{(m-1)2} & \dots & a_{(m-1)(n-1)} & a_{(m-1)n} \\ a_{m1} & a_{m2} & \dots & a_{m(n-1)} & a_{mn} \end{pmatrix}$$

c, b and A: prespecified.
x is vector of unknowns.
Solve LP for x.

LP in standard form.

(Matrix notation.)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b. \\ & x \geq 0. \end{aligned}$$

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

Slack Form

1. Next rewrite LP into ***slack form***.
2. Every inequality becomes equality.
3. All variables must be positive.
4. See resulting form on the right.

$$\begin{array}{rl} \max & c^T x \\ \text{subject to} & Ax = b. \\ & x \geq 0. \end{array}$$

1. New ***slack variables***. Rewrite inequality: $\sum_{i=1}^{n} a_i x_i \leq b$. As:

$$\begin{aligned} x_{n+1} &= b - \sum_{i=1}^{n} a_i x_i \\ x_{n+1} &\geq 0. \end{aligned}$$

2. Value of slack variable x_{n+1} encodes how far is the original inequality for holding with equality.

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Slack form...

1. LP now made of equalities of the form:
$$x_{n+1} = b - \sum_{i=1}^{n} a_i x_i$$
2. Variables on left: ***basic variables***.
3. Variables on right: ***nonbasic variables***.
4. LP in this form is in ***slack form***.

Linear program in slack form.

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \textit{for} \quad i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \ldots, n+m. \end{aligned}$$

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Slack form formally

Because everything is clearer when you use tuples. Not.

The slack form is defined by a tuple (N, B, A, b, c, v).

B - Set of indices of basic variables
N - Set of indices of nonbasic variables
$n = |N|$ - number of original variables
b, c - two vectors of constants
$m = |B|$ - number of basic variables
(i.e., number of inequalities)
$A = \{a_{ij}\}$ - The matrix of coefficients
$N \cup B = \{1, \ldots, n + m\}$
v - objective function constant.

Slack form formally

Final form

$$\begin{array}{ll} \text{Max} & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \textit{for} \quad i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \ldots, n+m. \end{array}$$

Example

Consider the following LP which is in slack form.

$$\begin{aligned}
\max \quad z &= 29 - \frac{1}{9}x_3 - \frac{1}{9}x_5 - \frac{2}{9}x_6 \\
x_1 &= 8 + \frac{1}{6}x_3 + \frac{1}{6}x_5 - \frac{1}{3}x_6 \\
x_2 &= 4 - \frac{8}{3}x_3 - \frac{2}{3}x_5 + \frac{1}{3}x_6 \\
x_4 &= 18 - \frac{1}{2}x_3 + \frac{1}{2}x_5
\end{aligned}$$

Example

...translated into tuple form (N, B, A, b, c, v).

$$B = \{1,2,4\}, N = \{3,5,6\}$$

$$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix} \qquad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/9 \\ -1/9 \\ -2/9 \end{pmatrix}$$

$$v = 29.$$

Note that indices depend on the sets N and B, and also that the entries in A are negation of what they appear in the slack form.

Another example...

$$\begin{aligned} \max \quad & 5x_1 + 4x_2 + 3x_3 \\ s.t. \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Transform into slack form...

$$\begin{aligned} \max \quad z &= \quad 5x_1 + 4x_2 + 3x_3 \\ s.t. \quad w_1 &= \quad 5 - 2x_1 - 3x_2 - x_3 \\ w_2 &= \quad 11 - 4x_1 - x_2 - 2x_3 \\ w_3 &= \quad 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

The Simplex algorithm by example

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\begin{array}{ll}
\max & 5x_1 + 4x_2 + 3x_3 \\
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\end{array}
$$

Next, we introduce slack variables, for example, rewriting $2x_1 + 3x_2 + x_3 \le 5$ as the constraints: $w_1 \ge 0$ and $w_1 = 5 - 2x_1 - 3x_2 - x_3$. The resulting LP in slack form is

$$
\Rightarrow \quad
\begin{array}{lrl}
\max & z = & 5x_1 + 4x_2 + 3x_3 \\
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1. w_1, w_2, w_3: slack variables. (Also currently basic variables).
2. Consider the slack representation trivial solution... all non-basic variables assigned zero: $x_1 = x_2 = x_3 = 0$.

1. $\implies w_1 = 5$, $w_2 = 11$ and $w_3 = 8$.
2. Feasible!
3. Objection function value: $z = 0$.
4. Further improve t value of objective function (i.e., z). While keeping feasibility.

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1. $x_1 = x_2 = x_3 = 0$ $\implies$ $w_1 = 5$, $w_2 = 11$ and $w_3 = 8$.
2. All w_i positive – change x_i a bit does not change feasibility.

1. $z = 5x_1 + 4x_2 + 3x_3$: want to increase values of x_1s... since z increases (since $5 > 0$).
2. How much to increase x_1???
3. Careful! Might break feasibility.
4. Increase x_1 as much as possible without breaking feasibility!

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$$\begin{aligned} \max \quad z = \;& 5x_1 + 4x_2 + 3x_3 \\ s.t. \quad w_1 = \;& 5 - 2x_1 - 3x_2 - x_3 \\ w_2 = \;& 11 - 4x_1 - x_2 - 2x_3 \\ w_3 = \;& 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

1. $x_1 = x_2 = x_3 = 0$ $\implies$ $w_1 = 5$, $w_2 = 11$ and $w_3 = 8$.
2. All w_i positive – change x_i a bit does not change feasibility.

1. $z = 5x_1 + 4x_2 + 3x_3$: want to increase values of x_1s... since z increases (since $5 > 0$).
2. How much to increase x_1???
3. Careful! Might break feasibility.
4. Increase x_1 as much as possible without breaking feasibility!

Example continued...

$$\begin{aligned} \max \quad z &= \quad 5x_1 + 4x_2 + 3x_3 \\ s.t. \quad w_1 &= \quad 5 - 2x_1 - 3x_2 - x_3 \\ w_2 &= \quad 11 - 4x_1 - x_2 - 2x_3 \\ w_3 &= \quad 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

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Example continued...

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\begin{aligned}
\max \quad z = &\quad 5x_1 + 4x_2 + 3x_3 \\
s.t. \quad w_1 = &\quad 5 - 2x_1 - 3x_2 - x_3 \\
w_2 = &\quad 11 - 4x_1 - x_2 - 2x_3 \\
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&x_1, x_2, x_3, w_1, w_2, w_3 \geq 0
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Example continued...

$$\begin{aligned} \max \quad z = & \quad 5x_1 + 4x_2 + 3x_3 \\ s.t. \quad w_1 = & \quad 5 - 2x_1 - 3x_2 - x_3 \\ w_2 = & \quad 11 - 4x_1 - x_2 - 2x_3 \\ w_3 = & \quad 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

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\begin{aligned}
\max \quad z = &\quad 5x_1 + 4x_2 + 3x_3\\
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w_2 = &\quad 11 - 4x_1 - x_2 - 2x_3\\
w_3 = &\quad 8 - 3x_1 - 4x_2 - 2x_3\\
&x_1, x_2, x_3, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

1. Set $x_2 = x_3 = 0$

$$
\begin{aligned}
w_1 &= 5 - 2x_1 - 3x_2 - x_3\\
&= 5 - 2x_1\\
w_2 &= 11 - 4x_1 - x_2 - 2x_3\\
&= 11 - 4x_1\\
w_3 &= 8 - 3x_1 - 4x_2 - 2x_3\\
&= 8 - 3x_1.
\end{aligned}
$$

1. Want to increase x_1 as much as possible, as long as:

$$
\begin{aligned}
w_1 &= 5 - 2x_1 \geq 0,\\
w_2 &= 11 - 4x_1 \geq 0,\\
\text{and } w_3 &= 8 - 3x_1 \geq 0.
\end{aligned}
$$

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ s.t. \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

1. Set $x_2 = x_3 = 0$

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Example continued...

$$\begin{aligned} \max \quad z = &\quad 5x_1 + 4x_2 + 3x_3 \\ s.t. \quad w_1 = &\quad 5 - 2x_1 - 3x_2 - x_3 \\ w_2 = &\quad 11 - 4x_1 - x_2 - 2x_3 \\ w_3 = &\quad 8 - 3x_1 - 4x_2 - 2x_3 \\ &x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

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Example continued...

$$\begin{aligned} \max \quad z &= 5x_1 + 4x_2 + 3x_3 \\ s.t. \quad w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\ w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\ w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

1. Constraints:

$$\begin{aligned} w_1 &= 5 - 2x_1 \geq 0, \\ w_2 &= 11 - 4x_1 \geq 0, \\ \text{and } w_3 &= 8 - 3x_1 \geq 0. \end{aligned}$$

2. $x_1 \leq 2.5$,
   $x_1 \leq 11/4 = 2.75$ and
   $x_1 \leq 8/3 = 2.66$

1. Maximum we can increase x_1 is 2.5.
2. $x_1 = 2.5, \; x_2 = 0, \; x_3 = 0, \; w_1 = 0, \; w_2 = 1, \; w_3 = 0.5$
   $$\Rightarrow \quad z = 5x_1 + 4x_2 + 3x_3 = 12.5.$$
3. Improved target!
4. A nonbasic variable x_1 is now non-zero. One basic variable (w_1) became zero.

Example continued...

1. Constraints:

$$w_1 = 5 - 2x_1 \geq 0,$$
$$w_2 = 11 - 4x_1 \geq 0,$$
and $w_3 = 8 - 3x_1 \geq 0.$

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$x_1 \leq 11/4 = 2.75$ and
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$$\begin{array}{rrl}
\max & z = & 5x_1 + 4x_2 + 3x_3 \\
s.t. & w_1 = & 5 - 2x_1 - 3x_2 - x_3 \\
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 & & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0
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Example continued...

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\begin{array}{rrl}
\max & z = & 5x_1 + 4x_2 + 3x_3 \\
s.t. & w_1 = & 5 - 2x_1 - 3x_2 - x_3 \\
& w_2 = & 11 - 4x_1 - x_2 - 2x_3 \\
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w_1 &= 5 - 2x_1 \geq 0, \\
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Example continued...

$$\begin{array}{rrcl} \max & z = & & 5x_1 + 4x_2 + 3x_3 \\ s.t. & w_1 & = & 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 & = & 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ & & & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{array}$$

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\begin{array}{rrl}
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Example continued...

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1. Want to keep invariant: All non-basic variables in current solution are zero...
2. Idea: Exchange x_1 and w_1!
3. Consider equality LP with w_1 and x_1.
   $w_1 = 5 - 2x_1 - 3x_2 - x_3$.
4. Rewrite as: $x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5\ x_3$.

Example continued...

$$\begin{array}{rrcl} \max & z = & & 5x_1 + 4x_2 + 3x_3 \\ s.t. & w_1 & = & 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 & = & 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ & & & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{array}$$

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Example continued...

$$\begin{aligned} \max \quad z &= 5x_1 + 4x_2 + 3x_3 \\ s.t. \quad w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\ w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\ w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

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Example continued...

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\end{aligned}$$

1. $x_1 = 2.5,\ x_2 = 0,\ x_3 = 0,\ w_1 = 0,\ w_2 = 1,\ w_3 = 0.5$
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Example continued...

Substituting $\mathbf{x_1 = 5 - 2x_1 - 3x_2 - x_3}$, the new LP

$$\begin{aligned}
\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3\\
x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3\\
w_2 &= 1 + 2w_1 + 5x_2\\
w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.
\end{aligned}$$

1. nonbasic variables: $\{w_1, x_2, x_3\}$
   basic variables: $\{x_1, w_2, w_3\}$.
2. Trivial solution: all nonbasic variables $= 0$ is feasible.
3. $w_1 = x_2 = x_3 = 0$. Value: $z = 12.5$.

Example continued...

Substituting $\mathbf{x_1 = 5 - 2x_1 - 3x_2 - x_3}$, the new LP

$$
\begin{aligned}
\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\
x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.
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Example continued...

Substituting $\mathbf{x_1 = 5 - 2x_1 - 3x_2 - x_3}$, the new LP

$$
\begin{aligned}
\max \quad z &= 12.5 - 2.5 w_1 - 3.5 x_2 + 0.5 x_3 \\
x_1 &= 2.5 - 0.5 w_1 - 1.5 x_2 - 0.5 x_3 \\
w_2 &= 1 + 2 w_1 + 5 x_2 \\
w_3 &= 0.5 + 1.5 w_1 + 0.5 x_2 - 0.5 x_3.
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Example continued...

1. Rewriting stop done is called ***pivoting***.
2. pivoted on x_1.
3. Continue pivoting till reach optimal solution.

$$\begin{aligned}
\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\
x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.
\end{aligned}$$

4. Can not pivot on w_1, since if w_1 increase, then z decreases. Bad.
5. Can not pivot on x_2 (coefficient in objective function is -3.5).
6. Can only pivot on x_3 since its coefficient ub objective 0.5. Positive number.

Example continued...

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$$
\begin{aligned}
\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\
x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
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$$\begin{aligned} \max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\ x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3. \end{aligned}$$

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Example continued...

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$$
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w_2 &= 1 + 2 w_1 + 5 x_2 \\
w_3 &= 0.5 + 1.5 w_1 + 0.5 x_2 - 0.5 x_3.
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 & w_2 = & 1 + 2w_1 \quad + 5x_2 \\
 & w_3 = & 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.
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\max & z = & 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\
& x_1 = & 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\
& w_2 = & 1 + 2w_1 \quad + 5x_2 \\
& w_3 = & 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.
\end{array}
$$

4. Can not pivot on w_1, since if w_1 increase, then z decreases. Bad.
5. Can not pivot on x_2 (coefficient in objective function is -3.5).
6. Can only pivot on x_3 since its coefficient ub objective 0.5. Positive number.

Example continued...

$$\begin{aligned}
\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3\\
x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3\\
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1. Can only pivot on x_3...
2. x_1 can only be increased to 1 before $w_3 = 0$.
3. Rewriting the equality for w_3 in LP: $w_3 = 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3$,
4. ...for x_3: $x_3 = 1 + 3w_1 + x_2 - 2w_3$.
5. Substituting into LP, we get the following LP.

$$\begin{aligned}
\max \quad z &= 13 - w_1 - 3x_2 - w_3\\
s.t. \quad x_1 &= 2 - 2w_1 - 2x_2 + w_3\\
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2. All coefficients in objective negative (or zero).
3. trivial solution (all nonbasic variables zero) is maximal.

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Pivoting changes nothing

Observation

Every pivoting step just rewrites the LP into EQUIVALENT LP.
When LP objective can no longer be improved because of rewrite, it implies that the original LP objective function can not be increased any further.

Simplex algorithm – summary

1. This was an informal description of the simplex algorithm.
2. At each step pivot on a nonbasic variable that improves objective function.
3. Till reach optimal solution.
4. Problem: Assumed that the starting (trivial) solution (all zero nonbasic vars) is feasible.

Starting somewhere...

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \qquad \forall i = 1, \ldots, n + m. \end{aligned}$$

1. L: Transformed LP to slack form.
2. **Simplex** starts from feasible solution and walks around till reaches opt.
3. L might not be feasible at all.
4. Example on left, trivial sol is not feasible, if $\exists b_i < 0$.

Idea: Add a variable x_0, and minimize it!

$$\begin{aligned} \min \quad & x_0 \\ \text{s.t.} \quad & x_i = x_0 + b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \ldots, n + m. \end{aligned}$$

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Finding a feasible solution...

1. $L' = \textbf{Feasible}(L)$ (see previous slide).
2. Add new variable x_0 and make it large enough.
3. $x_0 = \max(-\min_i b_i, 0)$, $\forall i > 0, x_i = 0$: feasible!
4. **LPStartSolution**(L'): Solution of **Simplex** to L'.
5. If x_0 is solution then L feasible, and we found a valid basic solution.
6. If $x_0 > 0$ then LP not feasible. Done.

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Lemma...

Lemma

LP L is feasible $\iff$ optimal objective value of LP L' is zero.

Proof.

A feasible solution to L is immediately an optimal solution to L' with $x_0 = 0$, and vice versa. Namely, given a solution to L' with $x_0 = 0$ we can transform it to a feasible solution to L by removing x_0. $\square$

Technicalities, technicalities everywhere

1. Starting solution for L', generated by **LPStartSolution**(L)..
2. .. not legal in slack form as non-basic variable x_0 assigned non-zero value.
3. Trick: Immediately pivoting on x_0 when running **Simplex**(L').
4. First try to decrease x_0 as much as possible.

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