CS 573: Algorithms, Fall 2013

Network Flow V - Min-cost flow

Lecture 16 October 22, 2013

Part I

Minimum Average Cost Cycle

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- **Q** $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: a *digraph*, *n* vertices, *m* edges.
- **2** $\omega: \mathbf{E} \to {\rm I\!R}$ weight on the edges.
- **3** directed cycle: closed walk $C = (v_0, v_1, \dots, v_t)$, where $v_t = v_0$ and $(v_i \rightarrow v_{i+1}) \in E$, for $i = 0, \dots, t-1$.
- average cost of a directed cycle is $\operatorname{AvgCost}(\mathsf{C}) = \omega(C) / t = (\sum_{e \in C} \omega(e)) / t.$
- **(5)** $d_k(v)$: min length of walk with exactly k edges, ending at v
- $\begin{tabular}{ll} \bullet & d_0(v)=0 \mbox{ and } & d_{k+1}(v)=\\ & \min_{e=(u\rightarrow v)\in E} \Bigl(d_k(u)+\omega(e) \Bigr) \,. \end{tabular}$
- Compute $d_i(v)$, for $\forall i, \forall v \in V$. In O(nm) time using dynamic programming.

Computing the Min-Average Cost cycle

Cost of *minimum average cost cycle* is $MinAvgCostCycle(\mathbf{G}) = \min_{\mathbf{C} \text{ is a cycle in } \mathbf{G}} AvgCost(\mathbf{C})$

Theorem

The minimum average cost of a directed cycle in **G** is equal to

$$lpha = \min_{v \in V} \max_{k=0}^{n-1} rac{d_n(v) - d_k(v)}{n-k}.$$

Namely, $\alpha = MinAvgCostCycle(G)$.

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Proof

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- Adding r to weight of every edge increases the average cost of a cycle AvgCost(C) by r.
- **2** α also increases by r.
- Solution \mathbf{O} Assume price of min. average cost cycle = \mathbf{O} .
- In all cycles have non-negative (average) cost.
- Prove: MinAvgCostCycle(G) = 0 $\implies \alpha = 0$. (Implies theorem by shifting prices by r).

Proof continued MinAvgCostCycle(G) = 0 $\implies \alpha \ge 0$

$$\textbf{0} \ \alpha = \min_{u \in V} \beta(u) \text{, where } \beta(u) = \max_{k=0}^{n-1} \frac{d_n(u) - d_k(u)}{n-k} .$$

- ② Assume lpha realized by vertex v; lpha=eta(v).
- ④ $oldsymbol{P}_n$: $oldsymbol{n}$ edges walk ending at $oldsymbol{v}$, of length $oldsymbol{d}_n(oldsymbol{v}).$
- ④ P_n must contain a cycle.
- **(**) Break P_n : a cycle π (length n k) and path σ (length k).

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Now, α = β(v) ≥ 0, by the choice of v.
QED for this direction.

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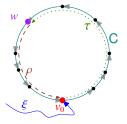


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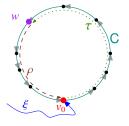
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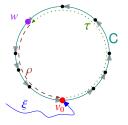
- $\min_{j=0}^{\infty} d_j(v_0)$ realized by index r < n.
 (Otherwise remove non-negative cycles.)
- ${}^{\textcircled{o}}$ ${m \xi}={}^{\textcircled{o}}$ walk of length r ending at v_0 .
- $w \in C$ = walk n r edges on C from v_0 .
- ${igsin 0} \,\, au$ is this walk (i.e., | au|=n-r).
- $\textcircled{0} \hspace{0.1in} d_n(w) \leq \omega \bigl(\xi \mid \mid \tau \bigr) = d_r(v_0) + \omega(\tau) \,,$
- $\bigcirc \rho$: walk on **C** from w back to v_0 .
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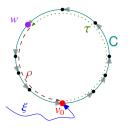
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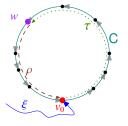
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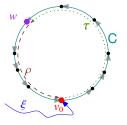
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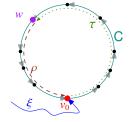
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- For any k: extend k edges shortest path ending at w to a path to v₀ (concatenating ρ)
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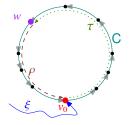


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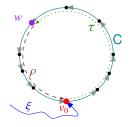
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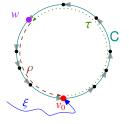
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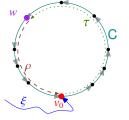
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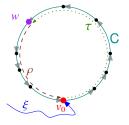
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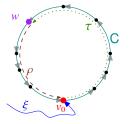
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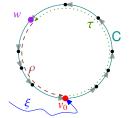
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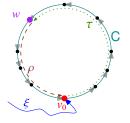
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$$\ \, {\mathfrak o} \ \, \alpha = \min_{v \in V({\mathsf G})} \beta(v) \leq \beta(w) \leq 0$$



QED

Computing α :

- $\forall k, \forall v \ d_k(v)$: longest path with k edges ending at v. Computed in O(nm) time.
- $a = \min_{v \in V} \max_{k=0}^{n-1} \frac{d_n(v) d_k(v)}{n-k}.$
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Finding min average cost cycle...

- Proved: Minimum avg cost of cycle in **G** is = $\alpha = \min_{v \in V} \max_{k=0}^{n-1} \frac{d_n(v) d_k(v)}{n-k}$.
- 2 Compute v that realizes α .
- (i) Add $-\alpha$ to all the edges in the graph.
- Looking for cycle of weight 0.
- **(5)** Recompute $d_i(\cdot)$ to agree with the new weights of the edges.
- \odot For v above: $0 = lpha = \max_{k=0}^{n-1} rac{d_n(v) d_k(v)}{n-k}$
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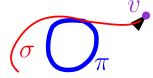
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- Repeat proof of theorem...
- ② P_n : path with n edges realizing $d_n(v)$.
- $\ \, {\boldsymbol S} \ \, P_n = \sigma || \pi$

- $\ \, \bullet \ \, \omega(\pi) \geq 0$
- ullet $\omega(\sigma) \geq d_k(v)$
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- π is a cycle and $\omega(\pi) = 0$. Done!
- Note the reweighting is not really necessary.

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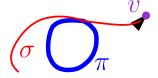
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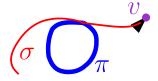
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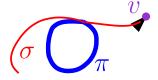
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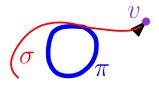
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 σ : a path of length k, π is a cycle.

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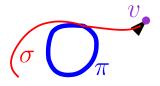


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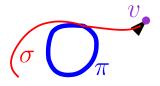
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Corollary

A direct graph **G** with *n* vertices and *m* edges, and a weight function $\omega(\cdot)$ on the edges, one can compute the cycle with minimum average cost in O(nm) time.

Part II

Potentials

Shortest path with negative weights...

- Dijkstra algorithm works only for graphs with non-negative weights.
- If negative weights, then one can use the Bellman-Ford algorithm.
- Sellman-Ford is slow... O(mn).
- Show how to use Dijkstra algorithm for some cases.
- **6** $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with weight $w(\cdot)$ on edges.
- $\mathbf{d}_{\omega}(s, t)$: length of shortest path.
- Weights might be negative!

Potential

A *potential* $p(\cdot)$ is a function that assigns a real value to each vertex of **G**, such that if $e = (u \rightarrow v) \in \mathbf{G}$ then $w(e) \ge p(v) - p(u)$.

Lemma

 $\exists p(\cdot) \text{ potential for } \mathsf{G} \iff \mathsf{G} \text{ has no negative cycles (for } w(\cdot)).$

Proof.

 \Rightarrow : Assume $\exists p(\cdot)$ potential. For any cycle **C**:

$$w(\mathsf{C}) = \sum_{(u
ightarrow v) \in \mathsf{E}(\mathsf{C})} w(e) \geq \sum_{(u
ightarrow v) \in E(\mathsf{C})} (p(v) - p(u)) = 0.$$

 \Leftarrow : Assume no negative cycle. p(v): shortest walk that ends at v. Claim: p(v) is a potential.

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 $\Leftarrow: \text{Assume no negative cycle. } p(v): \text{ shortest walk that ends at } v.$ Claim: p(v) is a potential.

• No negative cycles:
$$p(v)$$
 is well defined.
• $\forall (u \rightarrow v) \in \mathsf{E}(\mathsf{G})$: $p(v) \leq p(u) + w(u \rightarrow v)$
• $p(v) - p(u) \leq w(u \rightarrow v)$, as required.

Lemma

$$\begin{array}{l} p(\cdot): \mbox{ potential. } \forall e = (u \rightarrow v) \in \mathsf{E}(\mathsf{G}): \\ \ell(e) = w(e) - p(v) + p(u) \\ (A) \ \ell(\cdot) \ \mbox{ is non-negative for all edges.} \\ (B) \ \forall s, t \in \mathsf{V}(\mathsf{G}): \ \mbox{ shortest path } \pi \ \mbox{ of } \mathrm{d}_{\ell}(s,t) \ \mbox{ also s.p. } \mathrm{d}_{\omega}(s,t). \end{array}$$

$$\begin{array}{l} \text{Proof of (A):} \ w(e) \geq p(v) - p(u) \implies \\ w(e) - p(v) + p(u) \geq 0. \end{array}$$

$$\begin{array}{l} \text{Proof of (B):} \ \forall \ s - t \ \text{path} \ \pi \ \text{in } \mathbf{G}: \\ \ell(\pi) = \sum_{e = (u \rightarrow v) \in \pi} (w(e) - p(v) + p(u)) = \\ w(\pi) + p(s) - p(t), \\ \implies d_{\ell}(s,t) = d_{\omega}(s,t) + p(s) - p(t). \end{array}$$

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Lemma (iii)

Lemma

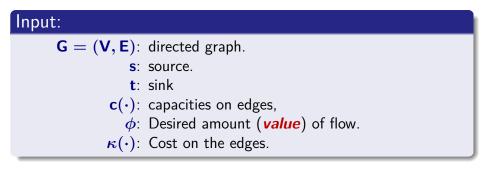
G: graph. $p(\cdot)$: potential. Compute the shortest path from s to all vertices of **G** in $O(n \log n + m)$ time, where **G** has n vertices and m edges

- Use Dijkstra algorithm on the distances defined by $\ell(\cdot)$.
- The shortest paths are preserved under this distance by Lemma (ii), and this distance function is always positive.

Part III

Minimum cost flow

Min cost flow



Definition - cost of flow

cost of flow f: $cost(f) = \sum_{e \in E} \kappa(e) * f(e)$.

Min-cost flow

minimum-cost s-t flow problem: compute the flow f of min cost that has value ϕ .

min-cost circulation problem

Instead of ϕ we have lower-bound $\ell(\cdot)$ on edges. (All flow that enters must leave.)

Claim

If we can solve min-cost circulation \implies can solve min-cost flow.

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Instead of ϕ we have lower-bound $\ell(\cdot)$ on edges. (All flow that enters must leave.)

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If we can solve min-cost circulation \implies can solve min-cost flow.

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Residual graph...

The *residual graph* of **f** is the graph $G_f = (V, E_f)$ where

$$E_{\mathsf{f}} = \left\{ e = (u \to v) \in V \times V \middle| \begin{array}{c} \mathsf{f}(e) < \mathsf{c}(e) \\ \text{or } \mathsf{f}(e^{-1}) > \ell(e^{-1}) \end{array} \right\}.$$

where
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Cost function is anti-symmetric:

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Some definitions

Definition

Cycle sign Directed cycle C in G_f .

$$e = (u \to v) \in \mathsf{E}(\mathsf{G}): \ \chi_C(e) = \begin{cases} 1 & e \in C \\ -1 & e^{-1} = (v \to u) \in C \\ 0 & \text{otherwise}; \end{cases}$$

Pay 1 if e is in C and -1 if we travel e in the "wrong" direction.

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Lemma

f, g: two valid circulations in G = (V, E). Let h = g - f. (A) h is a weak circulation, (B) if $h(u \rightarrow v) > 0$ then $(u \rightarrow v) \in G_f$.

Proof...

h is clearly a weak circulation (conservation of flow - verify).
If h(u → v) is negative, then h(v → u) = -h(u → v).
For e = (u → v), h(u → v) > 0:

(i) If e = (u → v) ∈ E, and f(e) < c(e) ⇒ e ∈ G_f
If f(e) = c(e) ⇒ h(e) = g(e) - f(e) ≤ 0.
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- ${\color{black} \bullet} \implies e^{-1} = (v \rightarrow u) \in {\color{black} \bullet}. \text{ Otherwise } {\color{black} \mathsf{h}}(u \rightarrow v) = 0.$
- $@ \ 0 > \mathsf{h} \, (e^{-1}) = \mathsf{g} \, (e^{-1}) \mathsf{f} \, (e^{-1}).$

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- Flow by **f** on e^{-1} larger than lower bound.
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