CS 573: Algorithms, Fall 2013

Network Flow V - Min-cost flow

Lecture 16
October 22, 2013

Part I

Minimum Average Cost Cycle

Minimum Average Cost Cycle

1. $\mathsf{G} = (\mathsf{V}, \mathsf{E})$: a ***digraph***, n vertices, m edges.
2. $\omega : \mathsf{E} \to \mathbb{R}$ weight on the edges.
3. ***directed cycle***: closed walk $\mathsf{C} = (v_0, v_1, \ldots, v_t)$, where $v_t = v_0$ and $(v_i \to v_{i+1}) \in \mathsf{E}$, for $i = 0, \ldots, t-1$.
4. ***average cost of a directed cycle*** is $\mathrm{AvgCost}(\mathsf{C}) = \omega(C)/t = \left(\sum_{e \in C} \omega(e)\right)/t$.
5. $d_k(v)$: min length of walk with exactly k edges, ending at v
6. $d_0(v) = 0$ and $d_{k+1}(v) = \min_{e=(u\to v)\in E}\bigl(d_k(u) + \omega(e)\bigr)$.
7. Compute $d_i(v)$, for $\forall i, \forall v \in \mathsf{V}$.
   In $O(nm)$ time using dynamic programming.

Computing the Min-Average Cost cycle

Cost of ***minimum average cost cycle*** is

$$\mathrm{MinAvgCostCycle}(\mathsf{G}) = \min_{\mathsf{C} \text{ is a cycle in } \mathsf{G}} \mathrm{AvgCost}(\mathsf{C})$$

Theorem

The minimum average cost of a directed cycle in G *is equal to*

$$\alpha = \min_{v \in V} \max_{k=0}^{n-1} \frac{d_n(v) - d_k(v)}{n-k}.$$

Namely, $\alpha = \mathrm{MinAvgCostCycle}(\mathsf{G})$.

Proof

Proof

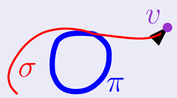
1. Adding r to weight of every edge increases the average cost of a cycle $\mathrm{AvgCost}(C)$ by r.
2. α also increases by r.
3. Assume price of min. average cost cycle $= 0$.
4. ... all cycles have non-negative (average) cost.
5. Prove: $\mathrm{MinAvgCostCycle}(G) = 0 \implies \alpha = 0$.
   (Implies theorem by shifting prices by r).

Proof continued

$\mathrm{MinAvgCostCycle}(G) = 0 \implies \alpha \geq 0$

Proof continued

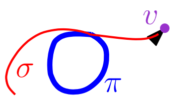
1. $\alpha = \min_{u \in V} \beta(u)$, where $\beta(u) = \max_{k=0}^{n-1} \dfrac{d_n(u) - d_k(u)}{n-k}$.
2. Assume α realized by vertex v; $\alpha = \beta(v)$.
3. P_n: n edges walk ending at v, of length $d_n(v)$.
4. P_n must contain a cycle.
5. Break P_n: a cycle π (length $n-k$) and path σ (length k).



6. $d_n(v) = \omega(P_n) = \omega(\pi) + \omega(\sigma) \geq \omega(\sigma) \geq d_k(v)$,

Proof continued

Continue proving: $\mathrm{MinAvgCostCycle}(G) = 0 \implies \alpha \geq 0$

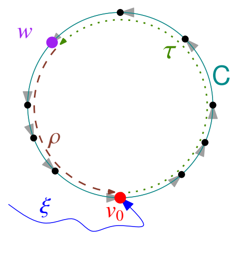


1. $\omega(\pi) \geq 0$: since π is cycle + by assumption $\forall$ cycle cost ≥ 0.
2. $\implies d_n(v) - d_k(v) \geq 0$. As such, $\frac{d_n(v) - d_k(v)}{n-k} \geq 0$. Let
   $$\beta(v) = \max_{j=0}^{n-1} \frac{d_n(v) - d_j(v)}{n-j} \geq \frac{d_n(v) - d_k(v)}{n-k} \geq 0.$$
   Now, $\alpha = \beta(v) \geq 0$, by the choice of v.
3. QED for this direction.

Proof for other direction

$\mathrm{MinAvgCostCycle}(G) = 0 \implies \alpha \leq 0$:

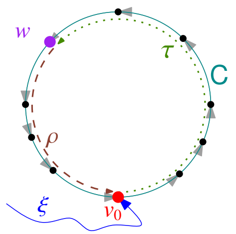
1. $C = (v_0, v_1, \ldots, v_t)$: directed cycle of weight 0.
2. $\min_{j=0}^{\infty} d_j(v_0)$ realized by index $r < n$. (Otherwise remove non-negative cycles.)
3. ξ = walk of length r ending at v_0.
4. $w \in C$ = walk $n - r$ edges on C from v_0.
5. τ is this walk (i.e., $|\tau| = n - r$).
6. $d_n(w) \leq \omega\big(\xi \,||\, \tau\big) = d_r(v_0) + \omega(\tau)$,
7. ρ: walk on C from w back to v_0.
8. $\tau \,||\, \rho$ goes around C several times.
9. $\omega(\tau \,||\, \rho) = 0$, as $\omega(C) = 0$.



Proof for other direction

$\mathrm{MinAvgCostCycle}(\mathbf{G}) = 0 \implies \alpha \leq 0$: continued

1. For any k: extend k edges shortest path ending at w to a path to v_0 (concatenating ρ)
2. $d_k(w) + \omega(\rho) \geq d_{k+|\rho|}(v_0) \geq d_r(v_0) \geq d_n(w) - \omega(\tau)$,
3. $\omega(\rho) \geq d_n(w) - \omega(\tau) - d_k(w)$.
4. $0 = \omega(\tau \,||\, \rho) = \omega(\rho) + \omega(\tau)$
   $\geq \big(d_n(w) - \omega(\tau) - d_k(w)\big) + \omega(\tau)$
   $= d_n(w) - d_k(w)$
5. $\implies \beta(w) = \max_{k=0}^{n-1} \frac{d_n(w) - d_k(w)}{n-k} \leq 0$.
6. $\alpha = \min_{v \in V(\mathbf{G})} \beta(v) \leq \beta(w) \leq 0$
7. $\implies \alpha = 0$.



QED

Computing α:

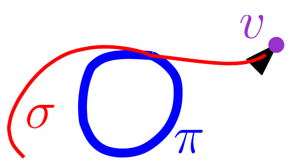
1. $\forall k, \forall v\ d_k(v)$: longest path with k edges ending at v. Computed in $O(nm)$ time.
2. $\alpha = \min_{v \in V} \max_{k=0}^{n-1} \frac{d_n(v) - d_k(v)}{n-k}$.
3. Compute α in $O(n^2)$ after $d_i(\cdot)$ computed.

Finding min average cost cycle...

1. Proved: Minimum avg cost of cycle in $\mathbf{G}$ is $=$ $\alpha = \min_{v \in V} \max_{k=0}^{n-1} \frac{d_n(v) - d_k(v)}{n-k}$.
2. Compute v that realizes α.
3. Add $-\alpha$ to all the edges in the graph.
4. Looking for cycle of weight 0.
5. Recompute $d_i(\cdot)$ to agree with the new weights of the edges.
6. For v above: $0 = \alpha = \max_{k=0}^{n-1} \frac{d_n(v) - d_k(v)}{n-k}$
7. $\implies \forall k \in \{0, \ldots, n-1\} \quad \frac{d_n(v) - d_k(v)}{n-k} \leq 0$
8. $\implies \forall k \in \{0, \ldots, n-1\} \quad d_n(v) - d_k(v) \leq 0$.
9. $\implies \forall i \qquad d_n(v) \leq d_i(v)$, for all i.

Finding min average cost cycle...

1. Repeat proof of theorem...
2. P_n: path with n edges realizing $d_n(v)$.
3. $P_n = \sigma || \pi$
   σ : a path of length k, π is a cycle.
4. $\omega(\pi) \geq 0$
5. $\omega(\sigma) \geq d_k(v)$
6. $\omega(\pi) = d_n(v) - \omega(\sigma) \leq d_n(v) - d_k(v) \leq 0$
7. π is a cycle and $\omega(\pi) = 0$. Done!
8. Note - the reweighting is not really necessary.



Finding min average cost cycle...

Corollary

A direct graph $\mathbf{G}$ with n vertices and m edges, and a weight function $\omega(\cdot)$ on the edges, one can compute the cycle with minimum average cost in $O(nm)$ time.

Part II

Potentials

Shortest path with negative weights...

1. Dijkstra algorithm works only for graphs with non-negative weights.
2. If negative weights, then one can use the Bellman-Ford algorithm.
3. Bellman-Ford is slow... $O(mn)$.
4. Show how to use Dijkstra algorithm for some cases.
5. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with weight $w(\cdot)$ on edges.
6. $\mathbf{d}_\omega(s, t)$: length of shortest path.
7. Weights might be negative!

Potential

A ***potential*** $p(\cdot)$ is a function that assigns a real value to each vertex of $\mathbf{G}$, such that if $e = (u \to v) \in \mathbf{G}$ then $w(e) \geq p(v) - p(u)$.

Lemma (i)

Lemma

$\exists p(\cdot)$ potential for $\mathbf{G}$ $\iff$ $\mathbf{G}$ has no negative cycles (for $w(\cdot)$).

Proof.

$\Rightarrow$: Assume $\exists p(\cdot)$ potential. For any cycle $\mathbf{C}$:

$$w(\mathbf{C}) = \sum_{(u \to v) \in \mathbf{E}(\mathbf{C})} w(e) \geq \sum_{(u \to v) \in E(\mathbf{C})} (p(v) - p(u)) = 0.$$

$\Leftarrow$: Assume no negative cycle. $p(v)$: shortest walk that ends at v.
Claim: $p(v)$ is a potential.

1. No negative cycles: $p(v)$ is well defined.
2. $\forall (u \to v) \in \mathbf{E}(\mathbf{G})$: $p(v) \leq p(u) + w(u \to v)$
3. $p(v) - p(u) \leq w(u \to v)$, as required.

□

Lemma (ii)

Lemma

$p(\cdot)$: potential. $\forall e = (u \to v) \in \mathbf{E}(\mathbf{G})$:
$\ell(e) = w(e) - p(v) + p(u)$
(A) $\ell(\cdot)$ is non-negative for all edges.
(B) $\forall s, t \in \mathbf{V}(\mathbf{G})$: shortest path π of $\mathbf{d}_\ell(s,t)$ also s.p. $\mathbf{d}_\omega(s,t)$.

Proof.

Proof of (A): $w(e) \geq p(v) - p(u) \implies$
$w(e) - p(v) + p(u) \geq 0.$

Proof of (B): $\forall$ $s - t$ path π in $\mathbf{G}$:
$\ell(\pi) = \sum_{e=(u\to v)\in\pi} (w(e) - p(v) + p(u)) =$
$w(\pi) + p(s) - p(t),$
$\implies \mathbf{d}_\ell(s,t) = \mathbf{d}_\omega(s,t) + p(s) - p(t).$

□

Lemma (iii)

Lemma

$\mathbf{G}$: graph. $p(\cdot)$: potential.
Compute the shortest path from s to all vertices of $\mathbf{G}$ in $O(n \log n + m)$ time, where $\mathbf{G}$ has n vertices and m edges

Proof.

1. Use Dijkstra algorithm on the distances defined by $\ell(\cdot)$.
2. The shortest paths are preserved under this distance by Lemma (ii), and this distance function is always positive.

□

Part III

Minimum cost flow

Min cost flow

Input:

$\mathbf{G} = (\mathbf{V}, \mathbf{E})$: directed graph.
$\mathbf{s}$: source.
$\mathbf{t}$: sink
$\mathbf{c}(\cdot)$: capacities on edges,
ϕ: Desired amount (***value***) of flow.
$\kappa(\cdot)$: Cost on the edges.

Definition - cost of flow

cost of flow $\mathbf{f}$: $\mathrm{cost}(\mathbf{f}) = \sum_{e \in E} \kappa(e) * \mathbf{f}(e)$.

Min cost flow problem

Min-cost flow

minimum-cost s-t flow problem: compute the flow $\mathbf{f}$ of min cost that has value ϕ.

min-cost circulation problem

Instead of ϕ we have lower-bound $\ell(\cdot)$ on edges.
(All flow that enters must leave.)

Claim

If we can solve min-cost circulation $\implies$ *can solve min-cost flow.*

HERE: All demands on vertices are zero!

Residual graph...

The ***residual graph*** of $\mathbf{f}$ is the graph $\mathbf{G_f} = (V, E_f)$ where

$$E_f = \left\{ e = (u \to v) \in V \times V \;\middle|\; \begin{array}{c} \mathbf{f}(e) < \mathbf{c}(e) \\ \text{or } \mathbf{f}\left(e^{-1}\right) > \ell\left(e^{-1}\right) \end{array} \right\}.$$

where $e^{-1} = (v \to u)$ if $e = (u \to v)$.

Assumption

$\forall u, v \quad (u \to v) \in \mathbf{E}(\mathbf{G}) \implies (v \to u) \notin \mathbf{E}(\mathbf{G})$.

Cost function is anti-symmetric:

$$\forall (u \to v) \in E_f \qquad \kappa\left((u \to v)\right) = -\kappa\left((v \to u)\right).$$

Some definitions

Definition

Cycle sign Directed cycle $\mathbf{C}$ in $\mathbf{G_f}$.

$$e = (u \to v) \in \mathbf{E}(\mathbf{G}): \ \chi_C(e) = \begin{cases} 1 & e \in C \\ -1 & e^{-1} = (v \to u) \in C \\ 0 & \text{otherwise;} \end{cases}$$

Pay $\mathbf{1}$ if e is in $\mathbf{C}$ and $-\mathbf{1}$ if we travel e in the "wrong" direction.

Definition

Cycle cost The ***cost*** of a directed cycle $\mathbf{C}$ in $\mathbf{G_f}$ is

$$\kappa(\mathbf{C}) = \sum_{e \in \mathbf{C}} \kappa(e).$$

Even more definitions

1. Circulation comply with capacity and lower-bounds constraints is ***valid***.
2. flow function that only comply with conservation property is a ***weak circulation***.
3. Weak circulation might violate capacity and lower bounds.
4. Weak circulation might not be a valid circulation.

Another lemma

Lemma

$\mathbf{f}, \mathbf{g}$: *two valid circulations in* $\mathbf{G} = (\mathbf{V}, \mathbf{E})$. *Let* $\mathbf{h} = \mathbf{g} - \mathbf{f}$.
(A) $\mathbf{h}$ *is a weak circulation,*
(B) if $\mathbf{h}(u \to v) > 0$ *then* $(u \to v) \in \mathbf{G_f}$.

Proof...

1. $\mathbf{h}$ is clearly a weak circulation (conservation of flow - verify).
2. If $\mathbf{h}(u \to v)$ is negative, then $\mathbf{h}(v \to u) = -\mathbf{h}(u \to v)$.
3. For $e = (u \to v)$, $\mathbf{h}(u \to v) > 0$:
   (i) If $e = (u \to v) \in E$, and $\mathbf{f}(e) < \mathbf{c}(e) \implies e \in \mathbf{G_f}$.
   If $\mathbf{f}(e) = \mathbf{c}(e) \implies \mathbf{h}(e) = \mathbf{g}(e) - \mathbf{f}(e) \leq 0$.
   Contradicts $\mathbf{h}(u \to v) > 0$.

Proof continued...

Proof continued:

For $e = (u \to v)$, $\mathbf{h}(u \to v) > 0$, and $e = (u \to v) \notin \mathbf{E}$:

1. $\implies e^{-1} = (v \to u) \in \mathbf{E}$. Otherwise $\mathbf{h}(u \to v) = 0$.
2. $0 > \mathbf{h}\left(e^{-1}\right) = \mathbf{g}\left(e^{-1}\right) - \mathbf{f}\left(e^{-1}\right)$.
3. $\implies \mathbf{f}\left(e^{-1}\right) > \mathbf{g}\left(e^{-1}\right) \geq \ell\left(e^{-1}\right)$.
4. Flow by $\mathbf{f}$ on e^{-1} larger than lower bound.
5. Can return this flow in the other direction.
6. $\implies e \in \mathbf{G_f}$. ■

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