

CS 573: Algorithms, Fall 2013

Homework 0, due Monday, September 2, 23:59:59, 2013

Name:	
Net ID:	

Neatly print your name (first name first, with no comma), your network ID, and a short alias into the boxes above. **Do not sign your name. Do not write your Social Security number.** Staple this sheet of paper to the top of your homework.

This homework tests your familiarity with the prerequisite material from CS 173, CS 225, and CS 373—many of these problems have appeared on homeworks or exams in those classes—primarily to help you identify gaps in your knowledge. **You are responsible for filling those gaps on your own.** Chapters 1–6 of CLR should be sufficient review, but you may want to consult other texts as well.

Before you do anything else, read the Homework Instructions and FAQ on the CS 573 course web page (<http://courses.engr.illinois.edu/cs573/fa2013/faq/>), and then check the box below. This web page gives instructions on how to write and submit homeworks—staple your solutions together in order, write your name and netID on every page, don't turn in source code, analyze everything, use good English and good logic, and so forth.

I have read the CS 573 Homework Instructions and FAQ.

Remember to do the quiz on moodle!

Version: 1.04

“Be that as it may, it is to night school that I owe what education I possess; I am the first to own that it doesn't amount to much, though there is something rather grandiose about the gaps in it.” – The tin drum, Gunter Grass

Required Problems

1. Moving numbers revisited. (60 PTS.)

(A) (30 PTS.) The input is a multiset X of n positive integer numbers. Consider the following famous algorithm:

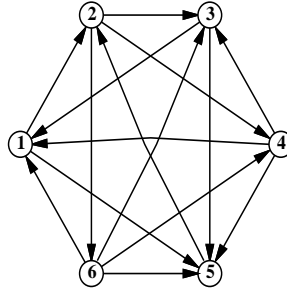
```
PlayItSam( $X$ ) :  
  while  $X$  contains more than two elements do  
    Three distinct elements  $x_1, x_2$  and  $x_3$  are chosen arbitrarily from  $X$ ,  
    such that  $x_1 \leq x_2 \leq x_3$   
    if  $x_1 = x_2$  then  
       $X \leftarrow (X \setminus \{x_1, x_2\}) \cup \{x_1 + x_2\}$   
      continue  
    if  $x_2 \geq x_3 - 1$  then  
       $X \leftarrow (X \setminus \{x_2, x_3\}) \cup \{x_2 + x_3\}$   
      continue  
     $X \leftarrow (X \setminus \{x_1, x_2, x_3\}) \cup \{x_1 + 1, x_2 + 1, x_3 - 2\}$ 
```

Prove (maybe using induction, but you do not have to) that **PlayItSam** always terminates.

- (B) (30 PTS.) (Harder.) Let $N = \sum_{x \in X} x$, and let $n = |X|$. Provide an upper bound, as tight as possible, using n and N on the running time of **PlayItSam**.

2. Snake in a tournament. (20 PTS.)

A **tournament** is a directed graph with exactly one edge between every pair of vertices. (Think of the nodes as players in a round-robin tournament, where each edge points from the winner to the loser.) A **Hamiltonian path** is a sequence of directed edges, joined end to end, that visits every vertex exactly once. Prove that every tournament contains at least one Hamiltonian path.



A six-vertex tournament containing the Hamiltonian path $6 \rightarrow 4 \rightarrow 5 \rightarrow 2 \rightarrow 3 \rightarrow 1$.

3. Some probability required. (20 PTS.)

There are n balls (numbered from 1 to n) and n boxes (numbered from 1 to n). We put each ball in a randomly selected box.

- (A) (4 PTS.) A box may contain more than one ball. Let S be the set of all the labels written on boxes that are not-empty. Let X be the smallest number in S . What is the expectation of X ?
- (B) (4 PTS.) What is the expected number of bins that have exactly one ball in them? (Hint: Compute the probability of a specific bin to contain exactly one ball and then use some properties of expectation.)
- (C) (8 PTS.) We put the balls into the boxes in such a way that there is exactly one ball in each box. If the number written on a ball is the same as the number written on the box containing the ball, we say there is a match. What is the expected number of matches?
- (D) (4 PTS.) What is the probability that there are exactly k matches? ($1 \leq k < n$)

[Hint: If you have to appeal to “intuition” or “common sense”, your answers are probably wrong!]