In this assignment, we will solve the PDE

$$-\nabla \cdot (p(\vec{x})\nabla u) = q(\vec{x}) \quad \text{for} \quad \vec{x} \in \Omega \subset \mathbb{R}^2,$$
$$u(\vec{x}) = g(\vec{x}) \quad \text{when} \quad \vec{x} \in \partial\Omega,$$

with

$$p(x,y) = x^2 + y^2$$

and

$$q(x,y) = \frac{\exp(y-x^2)}{x^2+y^2}\left(y^2 - 4x^2y^2 + 2y - 4x^4 - 3x^2\right)$$

and square domain, $\Omega = (-1,1) \times (-1,1)$ with a circular hole in the middle. You will be solving this PDE on a sequence of progressively refined meshes.

1. (a) Verify that the analytic solution to the PDE is

$$u(x,y) = \frac{\exp(y-x^2)}{x^2+y^2},$$

if $g = u$ on the boundary.

(b) On the previous homework assignment, we showed that

$$\iint_E p(\vec{x})\, \nabla\phi_r(\vec{x})^T \nabla\phi_s(\vec{x})\, d\vec{x}$$
$$= \left(J_T^{-T}\nabla\lambda_r\right)^T \left(J_T^{-T}\nabla\lambda_s\right) |J_T| \iint_S p(T(\vec{\alpha}))\, d\vec{\alpha}$$

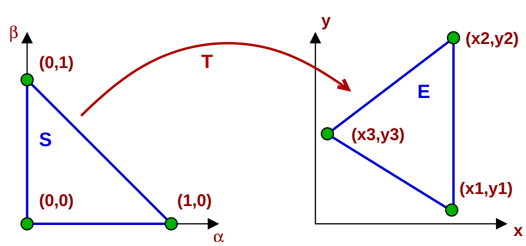
and

$$\iint_E q(\vec{x})\,\phi_r(\vec{x})\, d\vec{x} = |J_T| \iint_S q(T(\vec{\alpha}))\,\lambda_r(\vec{\alpha})\, d\vec{\alpha},$$

where E is a general triangle, S is the standard triangle, and T maps S to E. For this assignment, p and q are not piecewise constant, so we will approximate these integrals using midpoint quadrature. Show that

$$(x_c, y_c) = ([x_1 + x_2 + x_3]/3, [y_1 + y_2 + y_3]/3)$$

is the midpoint of E and that the midpoint of S is mapped to the midpoint of E.



(c) Show that

$$\frac{1}{2}p(x_c, y_c) = \iint_S p(x_c, y_c)\, d\vec{\alpha}$$

which implies that midpoint quadrature is equivalent to assuming p is piecewise constant on each triangle with value equal to p at the midpoint of the triangle.

(d) Show that

$$\frac{1}{2}q(x_c, y_c)\,\lambda_r(\alpha_c, \beta_c) = \iint_S q(x_c, y_c)\,\lambda_r(\vec{\alpha})\, d\vec{\alpha}$$

$$\text{and} \qquad \lambda_r(\alpha_c, \beta_c) = \frac{1}{3}$$

which implies that midpoint quadrature is equivalent to assuming q is piecewise constant on each triangle with value equal to q at the midpoint of the triangle.

(e) Write a routine that reads the files “elems#.dat”, “points#.dat”, and “bnd#.dat” into three arrays. The “points” array is of dimension $N \times 2$ where N is the number of vertices in the triangulation. The “elems” array is of dimension $M \times 4$ where M is the number of triangles. It contains the vertex numbers of each triangle in the first three columns and the fourth column indicates the subdomain number. The boundary array is of dimension $N_b \times 2$ where N_b is the number of vertices on the boundary. It contains the vertex numbers of the boundary points in the first column and the boundary number in the second column. Note that we are not using the domain or boundary numbers in this assignment. In Matlab you can use the command elems = load(['elems' num2str(pnum) '.dat']) to load the element data where pnum is an integer from one to four. Similar commands can be used to load the points and boundary data.

(f) On the last assignment, we assumed that p and q were piecewise constant. Modify your program from last assignment to use the value of p and q at the midpoint of each triangle in the assembly of the element “stiffness matrix” and element “load vector”.

(g) Modify your program from last assignment to use the list of boundary vertices in the first column of “bnd” to construct u^0. Use the analytic solution to set the value on the boundary. Make a similar modification for the part where you modified the global stiffness matrix and global load vector.

(h) Run your modified program and plot the solution on each of the four meshes.

(i) Plot the absolute value of the solution error for each of the four meshes.

(j) Plot the maximum of the error on each mesh as a function of $h_{\max}$ on a log scale. For each mesh, $h_{\max}$ is the maximum element radius. The radius of a triangle is the radius of its circumcircle:

$$r = \frac{\|\vec{x}_1 - \vec{x}_2\|\,\|\vec{x}_2 - \vec{x}_3\|\,\|\vec{x}_3 - \vec{x}_1\|}{2\,\|(\vec{x}_1 - \vec{x}_2) \times (\vec{x}_2 - \vec{x}_3)\|},$$

where the $\vec{x}_i$ are the vertices in $\mathbb{R}^3$. To use this formula in $\mathbb{R}^2$, assume the third coordinate is zero. What is the rate of convergence?