Homework 3 Assignment - Spring 2009

1. (30 pts) Consider the following hyperbolic PDE in conservation form,

$$\frac{\partial}{\partial t}u\left(x,t\right)+\frac{\partial}{\partial x}f\left(u\left(x,t\right)\right)=0,$$

where f is twice continuously differentiable. The Lax-Friedrichs method for this problem is

$$\frac{U_j^{n+1}-U_{j-1}^n+U_j^{n+1}-U_{j+1}^n}{2\Delta t}+\frac{F_{j+1}^n-F_{j-1}^n}{2\Delta x}=0,$$

where $F_j^n:=f\left(U_j^n\right)$.

   (a) Under what conditions is the truncation error $\mathcal{O}\left[\Delta t\right]+\mathcal{O}\left[\Delta x\right]$?

   (b) Compute the amplification factor, λ, for the Lax-Friedrichs method when it is applied to the linear problem $f\left(u\right)=a\,u$, where a is a constant. Hint: Let $U_j^n=\lambda^n\exp\left(i\,k\,x_j\right)$ and solve for λ. When is this method stable for this linear problem?

   (c) To write the Lax-Friedrichs method in the form needed for Ami Harten's TVD theorem (see book or notes), define

$$A_{j\pm\frac{1}{2}}^n:=\begin{cases}\dfrac{f\left(U_{j\pm1}^n\right)-f\left(U_j^n\right)}{\left(U_{j\pm1}^n-U_j^n\right)} & U_{j\pm1}^n\neq U_j^n\\[2ex] 0 & \text{otherwise}\end{cases},$$

   and determine the coefficients C_j and D_j used in the theorem. Is Lax-Friedrichs a TVD method?

2. (30 pts) Consider the invisid Burger's equation with homogeneous Dirichlet boundary conditions:

$$\frac{\partial}{\partial t}u\left(x,t\right)+\frac{\partial}{\partial x}\left(\frac{u\left(x,t\right)^2}{2}\right)=0,$$

$$0\le t\le t_f,\quad 0\le x\le 1,$$

$$u\left(x,0\right)=g\left(x\right),\quad u\left(0,t\right)=0,\quad u\left(1,t\right)=0.$$

Suppose the initial condition is given by

$$g\left(x\right)=\sin\left(2\pi\,x\right)+\frac{1}{2}\sin\left(\pi\,x\right)$$

   (a) In which direction are the characteristics pointing at $x=0$ and $x=1$ when $t=0$? Are the boundary conditions consistent with these characteristics?

   (b) Using the results from Exercise 4.3 in Morton & Mayers, at what point in space-time will the characteristic curves intersect?

   (c) Suppose t_f is small enough that the characteristic curves do not cross for $t\in\left[0,t_f\right]$. Show that the integral of the solution over space is conserved,

$$\frac{dI}{dt}=0,\quad\text{where}\quad I\left(t\right):=\int_0^1 u\left(x,t\right)dx.$$

   Using the result from derivations in class, is dI/dt still zero after a shock forms? What is the shock speed right after the shock forms?

   (d) Using trapozoidal rule in space, derive the following "numerical conservation equation":

$$\frac{\Delta I_n}{\Delta t}:=\frac{\Delta x}{\Delta t}\sum_{j=1}^{J-1}\left(U_j^{n+1}-U_j^n\right)\approx 0.$$

   (e) Compute $\Delta I_n/\Delta t$ for the Lax-Friedrichs method and the Roe Upwind method. For simplicity, you may assume that A_j^n never changes sign.

3. (40 pts) Write a program to numerically approximate the solution to the problem in Exercise 2, using four different methods:

   (M1) The standard upwind method

$$U_j^{n+1}=\begin{cases}U_j^n-\frac{\Delta t}{\Delta x}f'\left(U_j^n\right)\Delta_{+x}U_j^n, & f'\left(U_j^n\right)<0,\\[1ex] U_j^n-\frac{\Delta t}{\Delta x}f'\left(U_j^n\right)\Delta_{-x}U_j^n, & f'\left(U_j^n\right)\ge 0.\end{cases}$$

   (M2) The Lax-Wendroff method

$$U_j^{n+1}=U_j^n-\frac{\Delta t}{2\Delta x}\left(1-\frac{\Delta t}{\Delta x}A_{j+\frac{1}{2}}^n\right)\Delta_{+x}F_j^n-\frac{\Delta t}{2\Delta x}\left(1+\frac{\Delta t}{\Delta x}A_{j-\frac{1}{2}}^n\right)\Delta_{-x}F_j^n.$$

   (M3) The Roe upwind method

$$U_j^{n+1}=U_j^n-\gamma_{j,1}^n\,\frac{\Delta t}{\Delta x}\,\Delta_{+x}F_j^n-\gamma_{j,2}^n\,\frac{\Delta t}{\Delta x}\,\Delta_{-x}F_j^n.$$

$$\gamma_{j,1}^n=\frac{1-\operatorname{sgn}\left[A_{j+\frac{1}{2}}^n\right]}{2},\quad\gamma_{j,2}^n=\frac{1+\operatorname{sgn}\left[A_{j-\frac{1}{2}}^n\right]}{2},$$

$$\operatorname{sgn}(x)=x/\left|x\right|,\ \text{for}\ x\neq 0,\qquad \operatorname{sgn}(0)=0.$$

   (M4) The Lax-Friedrichs method

$$U_j^{n+1}=\frac{1}{2}\left(U_{j+1}^n+U_{j-1}^n\right)-\frac{\Delta t}{2\Delta x}\left(F_{j+1}^n-F_{j-1}^n\right)$$

   (a) Set $t_f=0.32$, and plot the solution at some interesting points in time both before and after the shock is formed. Set $\Delta t/\Delta x$ to be just small enough so that the maximum principle holds for the standard upwind method. Use the same $\Delta t/\Delta x$ for all four methods. Do you see any spurious oscillations?

(b) Plot $\Delta I_n / \Delta t$ as a function of time for all four methods. How well do the numerical methods respect the discrete conservation equation?

(c) Now, using the same Δx, select Δt to be 100 times smaller. Generate new plots of the solution at interesting points in time. Are the results better, or at least not worse, for all the methods? Can you explain what you observe for the Lax-Friedrichs method? Hint: What happens as $\Delta t \to 0$ for fixed Δx?

(d) Change the initial condition to

$$g(x) = \begin{cases} 1 & 0.2 < x < 0.5 \\ 0 & \text{otherwise} \end{cases},$$

and repeat your simulations, using relatively large but stable $\Delta t / \Delta x$. Plot the solution at a few interesting points. Which methods reproduce the correct behavior at the shock point? Which methods reproduce the correct behavior at the rarefaction point?