Homework 2 Assignment - Spring 2009

1. (Same as M&M Ex 3.5) The explicit method on a uniform mesh is used to solve the heat equation $u_t = u_{xx} + u_{yy}$ in the square region $0 \le x \le 1$, $0 \le y \le 1$. The boundary conditions specify that $u_x = 0$ on the side $x = 0$ of the square, and Dirichlet conditions are given on the rest of the boundary. At mesh points on $x = 0$ an additional line of mesh points with $x = -\Delta x$ is included, and the extra values $U^n_{-1,s}$ are then eliminated by use of the boundary condition. Show that the scheme becomes (at the points on the boundary $x = 0$),

$$\frac{U^{n+1}_{0,s} - U^n_{0,s}}{\Delta t} = \frac{2}{(\Delta x)^2}\left(U^n_{1,s} - U^n_{0,s}\right) + \frac{1}{(\Delta y)^2}\delta_y^2 U^n_{0,s}.$$

Show that the leading terms of the truncation error at this mesh point are

$$\begin{aligned} T^{n*}_{0,s} &= \frac{1}{2}\Delta t u_{tt} - \frac{1}{3}\Delta x u_{xxx} \\ &\quad - \frac{1}{12}\left[(\Delta x)^2 u_{xxxx} + (\Delta y)^2 u_{yyyy}\right] \end{aligned}$$

and deduce that if the usual stability condition is satisfied that the error in the solution satisfies

$$e^n_{r,s} \equiv U^n_{r,s} - u\left(x_r, y_s, t_n\right) = \mathcal{O}\left(\Delta t\right) + \mathcal{O}\left(\Delta x\right) + \mathcal{O}\left((\Delta y)^2\right).$$

2. (Same as M&M Ex 4.3) Verify that the function $u(x,t)$ defined implicity by the equation

$$u = f\left(x - ut\right)$$

is the solution of the problem

$$u_t + u\,u_x = 0, \qquad u\left(x, 0\right) = f\left(x\right),$$

and that $u(x,t)$ has the constant value of $f(x_0)$ on the straight line $x - x_0 = tf(x_0)$.

Show that the lines through the points $(x_0, 0)$ and $(x_0 + \epsilon, 0)$, where ϵ is small, meet at a point whos limit as $\epsilon \to 0$ is

$$\left(x_0 - f\left(x_0\right)/f'\left(x_0\right), -1/f'\left(x_0\right)\right).$$

Deduce that if $f'(x) \ge 0$ for all x the solution is single-valued for all positive t. More generally, show that if $f'(x)$ takes negative values, the solution $u(x,t)$ is single-valued for $0 \le t < t_c$, where $t_c = -1/M$ and $M \le f'(x)$.

Show that for the function

$$f(x) = \exp\left[-10\left(4x - 1\right)^2\right]$$

the critical value is $t_c = \exp\left(\frac{1}{2}\right)/\left(8\sqrt{5}\right)$ which is about 0.092.

3. Consider the heat-equation

$$u_t = u_{xx} + 2u_{yy}$$

on the domain

$$0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le t \le t_f$$

with initial conditions

$$u\left(x, y, 0\right) = \begin{cases} 5 & \frac{1}{16} \le \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \le \frac{1}{8} \\ 0 & \text{otherwise} \end{cases}$$

and boundary conditions

$$\frac{\partial u}{\partial x}\left(0, y, t\right) = 0, \quad \frac{\partial u}{\partial y}\left(x, 0, t\right) = 0,$$

$$u\left(x, 1, t\right) = 0, \quad u\left(1, y, t\right) = 0.$$

(a) Apply the "Crank-Nicolson Method" using the additional grid point method described in the first problem to treat the Neumann boundary conditions along the $x = 0$ and $y = 0$ boundaries. You can use Matlab's "backslash" operator to solve the resulting linear systems of equations.

(b) If you have not done so already, change your program to "pre-factor" the matrix used in the linear solve at each time step. In other words, compute the LU factorization before starting the time stepping process, then use these LU factors in each step. Plot the sparsity pattern of the unfactored matrix and its LU factors. Describe the bandwidth of the matrix and the bandwidth of the LU factors in terms of the number of grid points in the x and y directions.

(c) Plot the numerical solution at $t = 0$ and a couple other interesting points in the future. How long before maximum value of the numerical solution is less than 10^{-3} in magnitude?

(d) Show what happens when you use a stepsize combination that violates the conditions for the numerical maximum principle.