Homework 1 Assignment - Spring 2009

1. Verify that

$$\langle \psi_m, \psi_n \rangle = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases},$$

where $\psi_m(x) = \sin(m\pi x)$ and

$$\langle f, g \rangle := 2 \int_0^1 f(x)\, g(x)\, dx.$$

2. Consider the "weighted average" method

$$y^{n+1} = y^n + \Delta t\, \theta\, f\left(y^{n+1}\right) + \Delta t\, (1-\theta)\, f(y^n)$$

for approximating the solution to first-order ODEs of the form

$$\frac{dy}{dt} = f(y), \quad \text{with} \quad y(0) = c.$$

(a) Derive a bound on the error in "one step" of the method when applied to the linear, scalar problem

$$\frac{dy}{dt} = \alpha y, \quad \text{with} \quad y(0) = c.$$

Express this bound in the form

$$\left|y^1 - y(\Delta t)\right| \leq C_1 \Delta t^{k+1},$$

where k and C_1 are constants, and Δt sufficiently small. You may find it useful to use $(1-x)^{-1} = 1 + x + x^2 + \cdots$ for $|x| < 1$.

(b) What is k when $\theta = 1/2$? What is k when $0 \leq \theta < 1/2$ or $1/2 < \theta \leq 1$?

(c) Show that the n-step error can be bounded by

$$|y^n - y(n\Delta t)| \leq C_2 \Delta t^k,$$

when $n\Delta t$ is bounded by a final time, t_f, where C_2 is a constant.

3. (Same as M&M Ex 2.1)

(a) The function $u^0(x)$ is defined on $[0,1]$ by

$$u^0(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x \leq 1/2, \\ 2 - 2x & \text{if} \quad 1/2 \leq x \leq 1. \end{cases}$$

Show that

$$u^0(x) = \sum_{m=1}^{\infty} a_m \sin(m\pi x).$$

where

$$a_m = \frac{8}{m^2\pi^2} \sin\left(\frac{m\pi}{2}\right).$$

(b) Show that

$$\int_{2p}^{2p+2} \frac{1}{x^2} dx > \frac{2}{(2p+1)^2}$$

and hence that

$$\sum_{p=p_0}^{\infty} \frac{1}{(2p+1)^2} < \frac{1}{4p_0}.$$

(c) Deduce that $u^0(x)$ is approximated on the interval $[0,1]$ to within 0.001 by the sine series above truncated after $m = 405$.

4. (Similar to M&M Ex 2.2i)

(a) Show that for every positive value of $\mu = \Delta t / (\Delta x)^2$ there exists a constant $C(\mu)$ such that, for all positive values of k and Δx,

$$\left|1 - 4\mu \sin^2\left(\frac{1}{2} k\Delta x\right) - e^{-k^2 \Delta t}\right| \leq C(\mu)\, k^4 (\Delta t)^2.$$

Verify that when $\mu = 1/4$ this inequality is satisfied by $C = 5/6$.

(b) Define the function

$$g(k\Delta x) := \left|1 - 4\mu \sin^2\left(\frac{1}{2} k\Delta x\right) - \exp\left[-(k\Delta x)^2 \mu\right]\right|$$

for $\mu = 1/4$, and plot $g(z)/\left(\mu^2 z^4\right)$ on the interval $0 < z \leq 4$. What can you conclude about the "sharpness" of the bound of $C(1/4) \leq 5/6$?

(c) Bonus: Prove that $C(1/4) \leq 1/2$.

5. Write a computer program to solve the heat equation on the domain $x \in [0,1]$, $t \in [0, t_F]$ using forward-difference in time and centered difference in space. Assume Dirichlet boundary conditions and $t_F = 0.2$.

(a) Test the stability of your algorithm using zero Dirichlet boundary conditions and the initial conditions specified in problem 3. Try a few difference values of Δt and Δx above and below (but near) the stability threshold. Numerically verify the stability threshold. For each $\Delta t, \Delta x$ pair make a single 2d plot of the solution at several fixed points in time (i.e., similar to the plots in the lecture notes).

(b) Test the accuracy of your algorithm. At what point in space-time is the largest absolute error committed, what about relative error? Plot the absolute and relative error as a function of space and time for a stable simulation. Make sure your plot shows something useful (you may need to plot the log of the error or use a larger stepsize). Compare your numerical solution to a suitably the truncated analytic solution (i.e., use the results of problem 3).

(c) Experiment with the problem by adding time-dependent boundary conditions (e.g., $u(0,t) = b\sin(\omega t)$) or some interesting source term. Plot the results in space-time as 2d plots or a 3d surface.