# Structure from Motion

Computer Vision

CS 543 / ECE 549

University of Illinois

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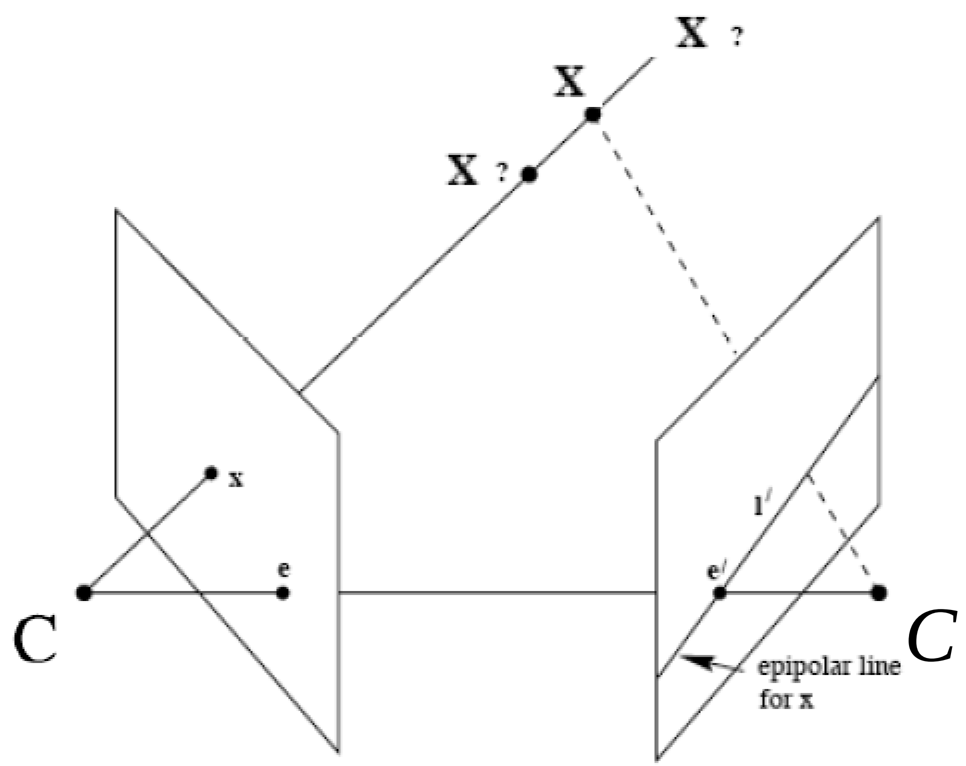
Many slides adapted from Lana Lazebnik, Silvio Saverese, Steve Seitz, Martial Hebert

# This class: structure from motion

- Recap of epipolar geometry
  - Depth from two views
- Projective structure from motion
- Affine structure from motion

# Recap: Epipoles

- Point x in left image corresponds to **epipolar line** l’ in right image
- Epipolar line passes through the epipole (the intersection of the cameras’ baseline with the image plane



# Recap: Fundamental Matrix

- Fundamental matrix maps from a point in one image to a line in the other

$$\mathbf{l}' = \mathbf{F}\mathbf{x} \qquad \mathbf{l} = \mathbf{F}^\top \mathbf{x}'$$

- If x and x’ correspond to the same 3d point X:

$$\mathbf{x}'^\top \mathbf{F}\mathbf{x} = 0$$

# Recap: Automatic Estimation of F

Assume we have matched points x ⇔ x' with outliers

**8-Point Algorithm for Recovering F**

- Correspondence Relation

$$\mathbf{x}'^T \mathbf{F} \mathbf{x} = 0$$

1. Normalize image coordinates

$$\tilde{\mathbf{x}} = \mathbf{T}\mathbf{x} \quad \tilde{\mathbf{x}}' = \mathbf{T}'\mathbf{x}'$$

2. RANSAC with 8 points
   - Randomly sample 8 points
   - Compute F via least squares
   - Enforce $\det\left(\tilde{\mathbf{F}}\right) = 0$ by SVD
   - Repeat and choose F with most inliers
3. De-normalize: $\mathbf{F} = \mathbf{T}'^T \tilde{\mathbf{F}} \mathbf{T}$

# Recap

- We can get projection matrices P and P' up to a projective ambiguity (see HZ p. 255-256)

$$\mathbf{P} = [\mathbf{I} \mid \mathbf{0}] \quad \mathbf{P}' = [[\mathbf{e}']_{\times}\mathbf{F} \mid \mathbf{e}'] \quad \mathbf{e}'^{T}\mathbf{F} = 0$$

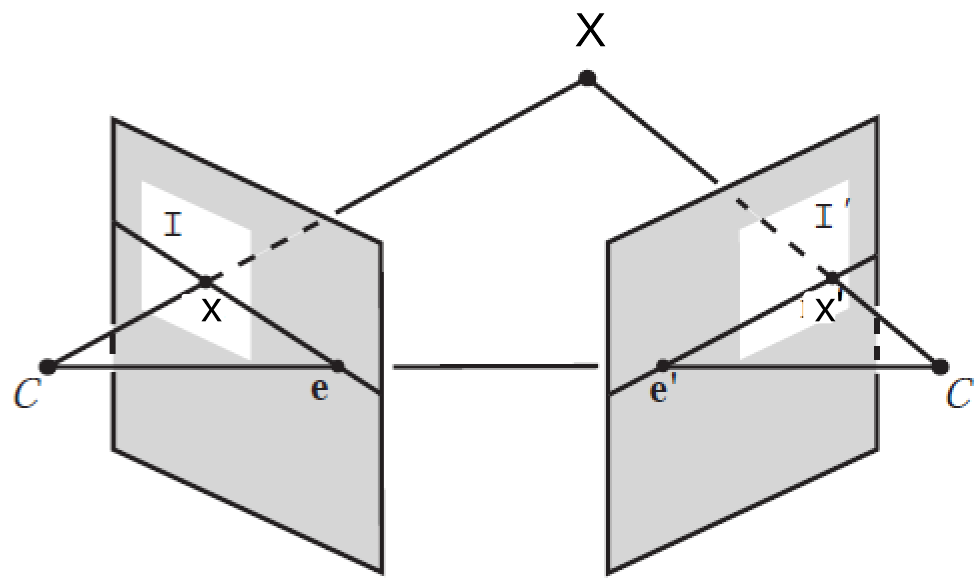
See HZ p. 255-256

- Code:

```
function P = vgg_P_from_F(F)
[U,S,V] = svd(F);
e = U(:,3);
P = [-vgg_contreps(e)*F e];
```

# Triangulation: Linear Solution

- Generally, rays C→x and C'→x' will not exactly intersect
- Can solve via SVD, finding a least squares solution to a system of equations




$$\mathbf{x} = \mathbf{P}\mathbf{X} \qquad \mathbf{x}' = \mathbf{P}'\mathbf{X}$$

$$\mathbf{A}\mathbf{X} = \mathbf{0} \qquad \mathbf{A} = \begin{bmatrix} u\mathbf{p}_3^T - \mathbf{p}_1^T \\ v\mathbf{p}_3^T - \mathbf{p}_2^T \\ u'\mathbf{p}_3'^T - \mathbf{p}_1'^T \\ v'\mathbf{p}_3'^T - \mathbf{p}_2'^T \end{bmatrix}$$

Further reading: HZ p. 312-313

# Triangulation: Linear Solution

Given **P**, **P'**, **x**, **x'**

1. Precondition points and projection matrices
2. Create matrix **A**
3. [U, S, V] = svd(A)
4. **X** = V(:, end)

$$\mathbf{x} = w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad \mathbf{x}' = w\begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1^T \\ \mathbf{p}_2^T \\ \mathbf{p}_3^T \end{bmatrix} \quad \mathbf{P}' = \begin{bmatrix} \mathbf{p}_1'^T \\ \mathbf{p}_2'^T \\ \mathbf{p}_3'^T \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} u\mathbf{p}_3^T - \mathbf{p}_1^T \\ v\mathbf{p}_3^T - \mathbf{p}_2^T \\ u'\mathbf{p}_3'^T - \mathbf{p}_1'^T \\ v'\mathbf{p}_3'^T - \mathbf{p}_2'^T \end{bmatrix}$$

Pros and Cons

- Works for any number of corresponding images
- Not projectively invariant

Code: http://www.robots.ox.ac.uk/~vgg/hzbook/code/vgg_multiview/vgg_X_from_xP_lin.m

# Triangulation: Non-linear Solution

- Minimize projected error while satisfying $\hat{\boldsymbol{x}}'^{T}\boldsymbol{F}\hat{\boldsymbol{x}}=0$

$$cost(\boldsymbol{X}) = dist(\boldsymbol{x},\hat{\boldsymbol{x}})^2 + dist(\boldsymbol{x}',\hat{\boldsymbol{x}}')^2$$

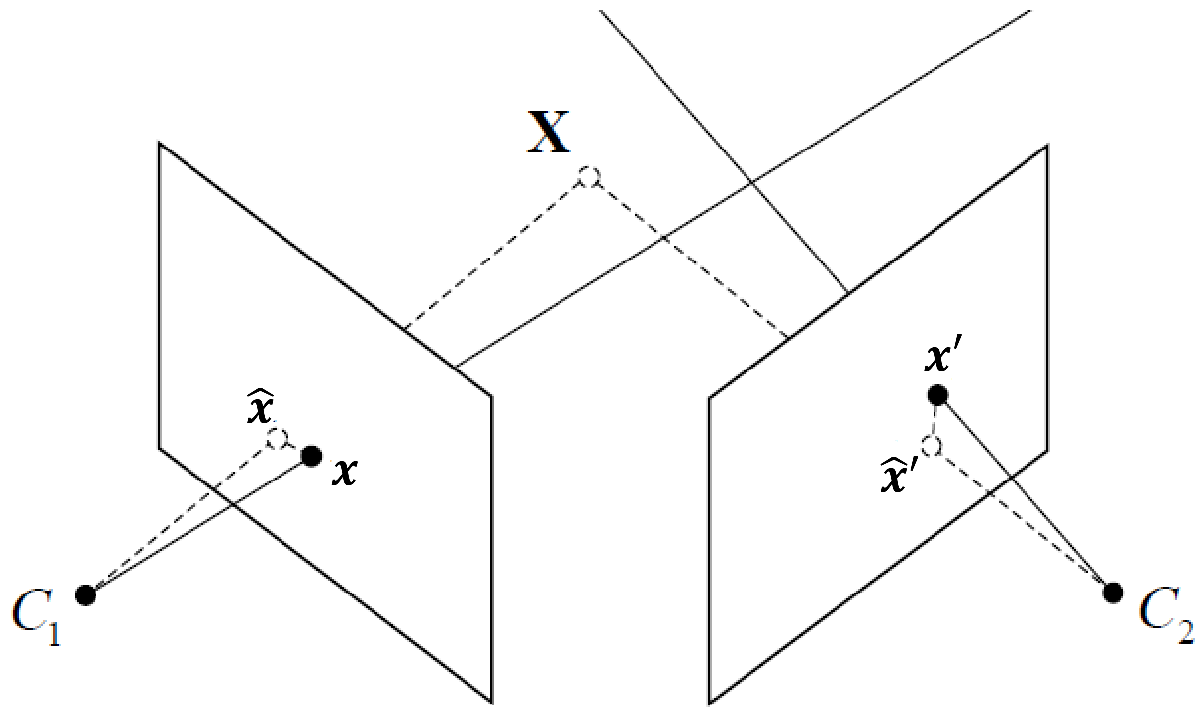



Figure source: Robertson and Cipolla (Chpt 13 of Practical Image Processing and Computer Vision)

# Triangulation: Non-linear Solution

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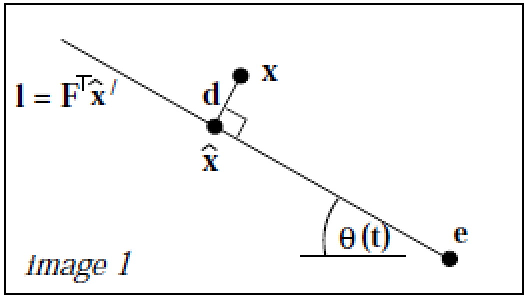
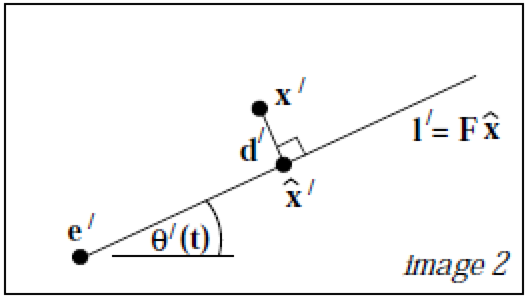


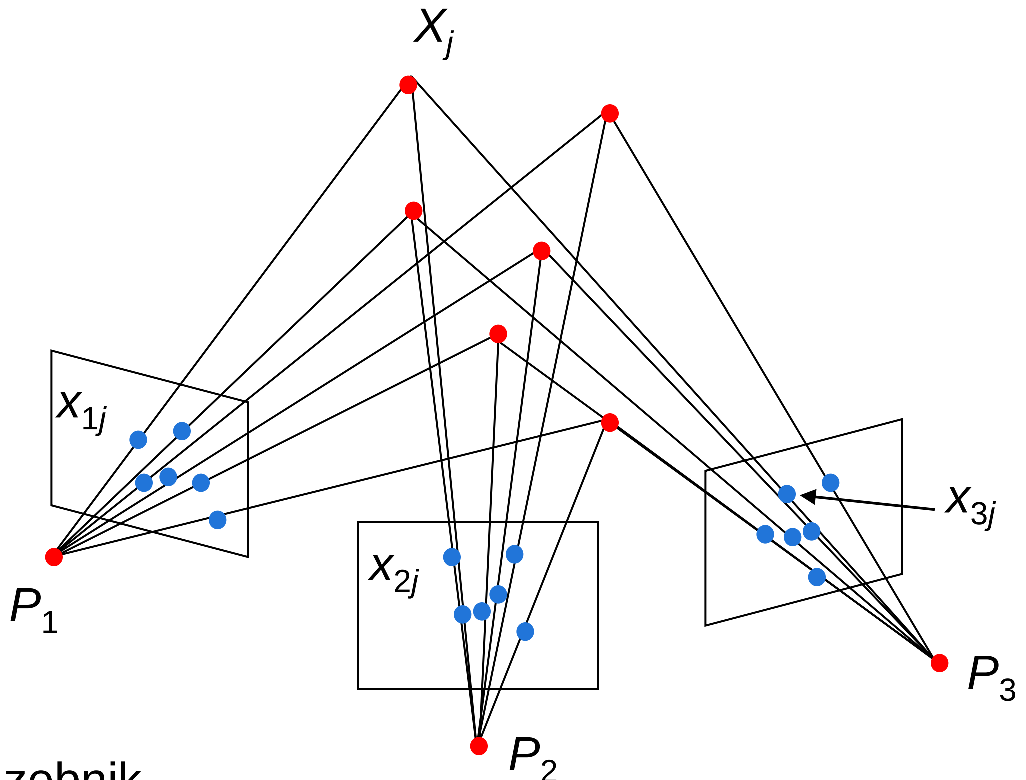




- Solution is a 6-degree polynomial of *t*, minimizing $d(\mathbf{x}, \mathbf{l}(t))^2 + d(\mathbf{x}', \mathbf{l}'(t))^2$

Further reading: HZ p. 318

# Projective structure from motion

- Given: *m* images of *n* fixed 3D points
  - $\mathbf{x}_{ij} = \mathbf{P}_i \mathbf{X}_j\,,\ i = 1,\ldots, m,\quad j = 1, \ldots, n$
- Problem: estimate *m* projection matrices $\mathbf{P}_i$ and *n* 3D points $\mathbf{X}_j$ from the *mn* corresponding 2D points $\mathbf{x}_{ij}$




Slides from Lana Lazebnik

# Projective structure from motion

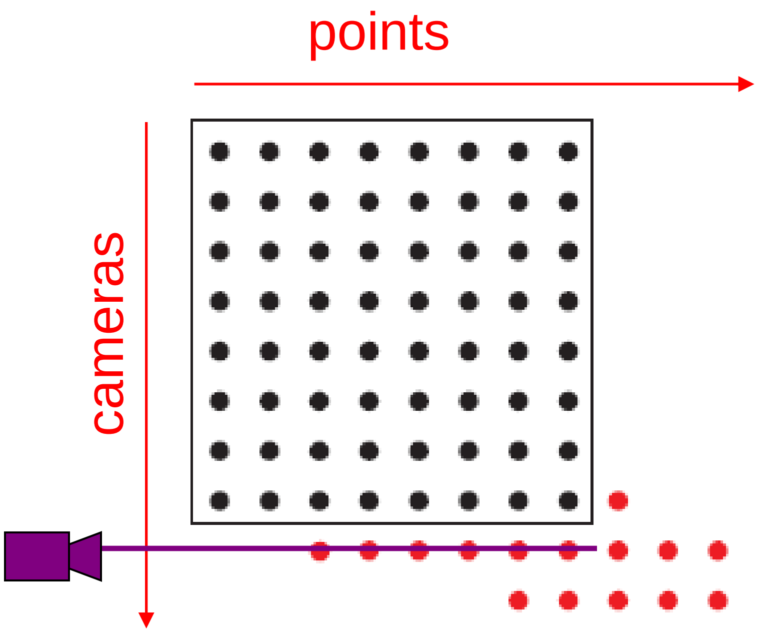
- Given: *m* images of *n* fixed 3D points
  - $\mathbf{x}_{ij} = \mathbf{P}_i \mathbf{X}_j, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n$
- Problem: estimate *m* projection matrices $\mathbf{P}_i$ and *n* 3D points $\mathbf{X}_j$ from the *mn* corresponding points $\mathbf{x}_{ij}$
- With no calibration info, cameras and points can only be recovered up to a 4x4 projective transformation **Q**:
  - $\mathbf{X} \rightarrow \mathbf{QX}, \mathbf{P} \rightarrow \mathbf{PQ}^{-1}$
- We can solve for structure and motion when
  - $2mn \geq 11m + 3n - 15$
- For two cameras, at least 7 points are needed

# Sequential structure from motion

- Initialize motion from two images using fundamental matrix

- Initialize structure by triangulation

- For each additional view:
  - Determine projection matrix of new camera using all the known 3D points that are visible in its image – *calibration*



# Sequential structure from motion

- Initialize motion from two images using fundamental matrix

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- For each additional view:
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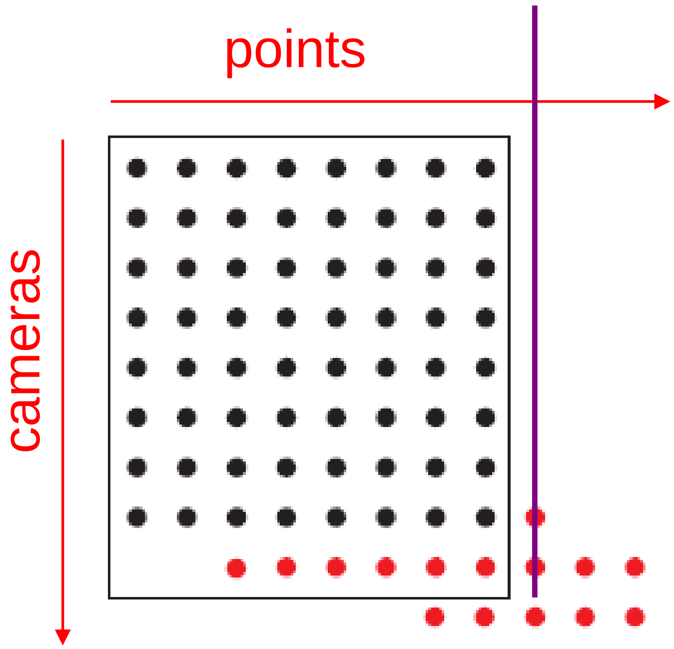
  - Refine and extend structure: compute new 3D points, re-optimize existing points that are also seen by this camera – *triangulation*



# Sequential structure from motion

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- For each additional view:
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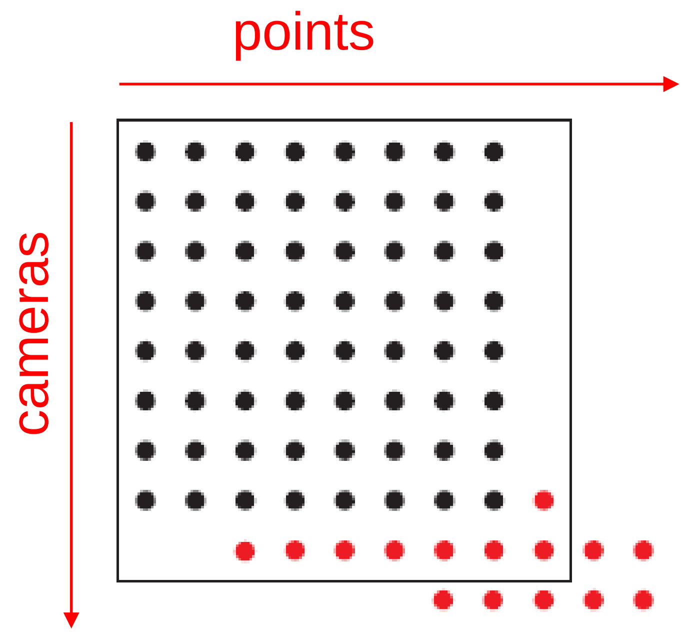
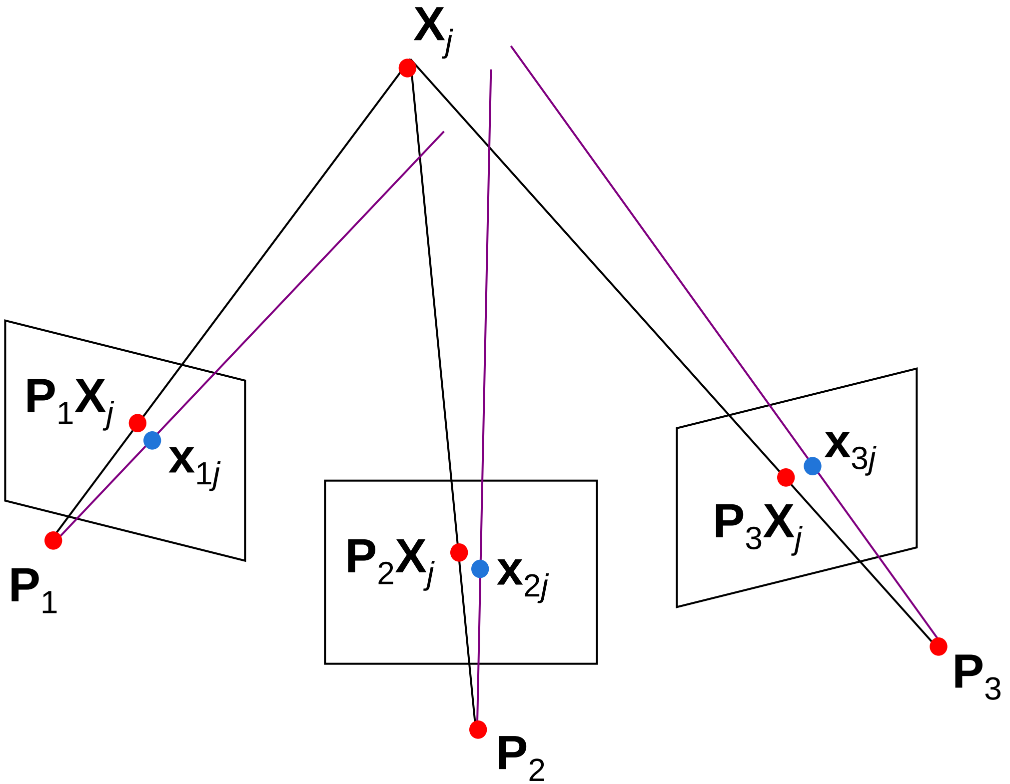



- Refine structure and motion: bundle adjustment

# Bundle adjustment

- Non-linear method for refining structure and motion
- Minimizing reprojection error

$$E(\mathbf{P},\mathbf{X}) = \sum_{i=1}^{m}\sum_{j=1}^{n} D\left(\mathbf{x}_{ij}, \mathbf{P}_i \mathbf{X}_j\right)^2$$



# Auto-calibration

- Auto-calibration: determining intrinsic camera parameters directly from uncalibrated images

- For example, we can use the constraint that a moving camera has a fixed intrinsic matrix
  - Compute initial projective reconstruction and find 3D projective transformation matrix $\mathbf{Q}$ such that all camera matrices are in the form $\mathbf{P}_i = \mathbf{K}\,[\mathbf{R}_i \mid \mathbf{t}_i]$

- Can use constraints on the form of the calibration matrix, such as zero skew

# Summary so far

- From two images, we can:
  - Recover fundamental matrix F
  - Recover canonical cameras P and P' from F
  - Estimate 3D positions (if K is known) that correspond to each pixel
- For a moving camera, we can:
  - Initialize by computing F, P, X for two images
  - Sequentially add new images, computing new P, refining X, and adding points
  - Auto-calibrate assuming fixed calibration matrix to upgrade to similarity transform

# Photo synth

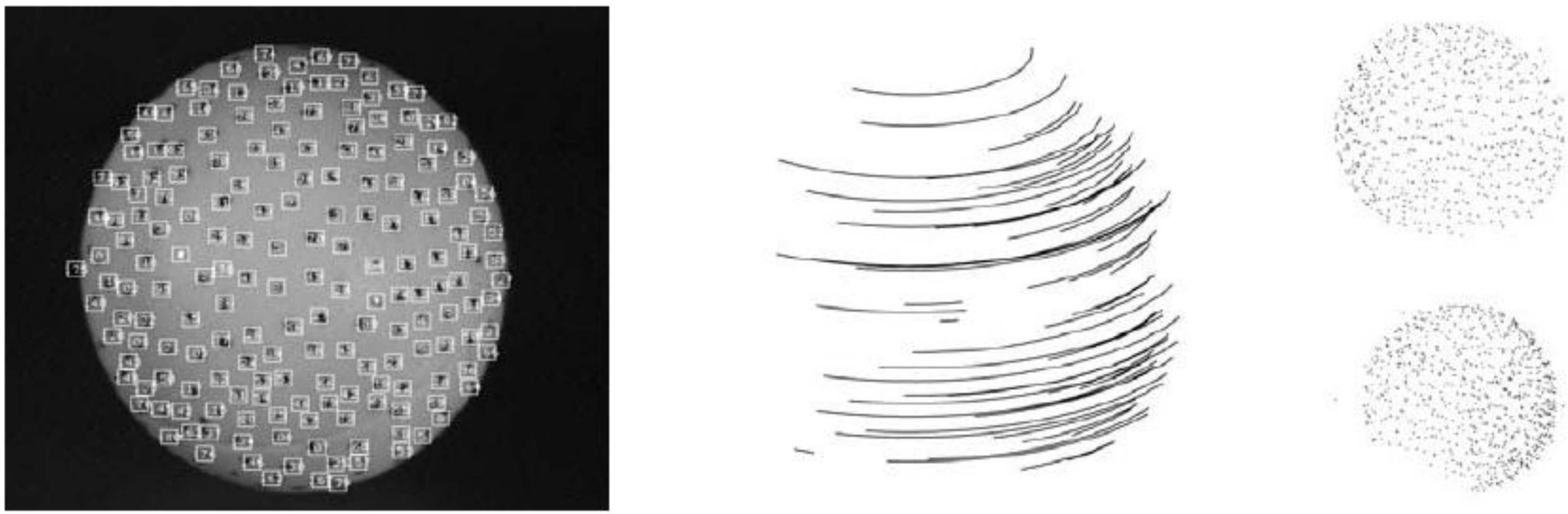
Noah Snavely, Steven M. Seitz, Richard Szeliski, "Photo tourism: Exploring photo collections in 3D," SIGGRAPH 2006

http://photosynth.net/

# 3D from multiple images

Building Rome in a Day: Agarwal et al. 2009

# Structure from motion under orthographic projection



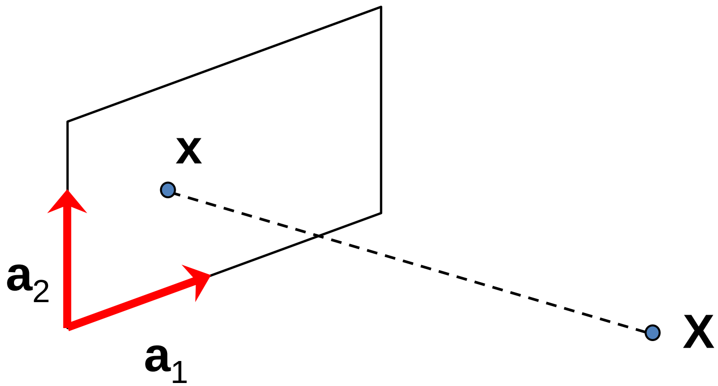
(a) (b) (c)

3D Reconstruction of a Rotating Ping-Pong Ball

- Reasonable choice when
  - Change in depth of points in scene is much smaller than distance to camera
  - Cameras do not move towards or away from the scene

C. Tomasi and T. Kanade. [Shape and motion from image streams under orthography: A factorization method.](#) *IJCV*, 9(2):137-154, November 1992.

# Orthographic projection for rotated/translated camera




$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

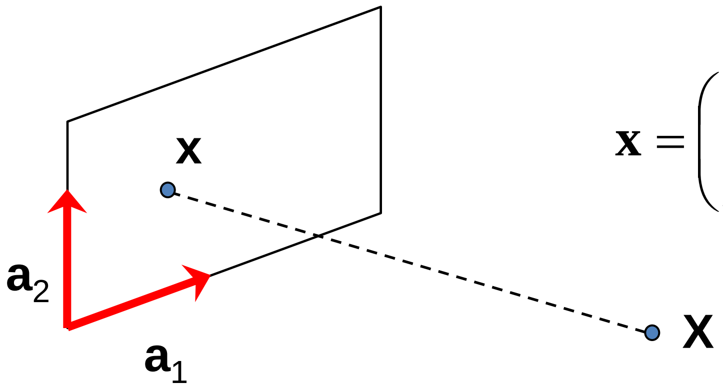
$$\begin{pmatrix} u_{fp} \\ v_{fp} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \left( R'_f \begin{bmatrix} X_p \\ Y_p \\ Z_p \end{bmatrix} + t_f \right)$$

$$R_f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} R'_f$$

$$\begin{pmatrix} u_{fp} \\ v_{fp} \end{pmatrix} = R_f \begin{bmatrix} X_p \\ Y_p \\ Z_p \end{bmatrix} + t_f$$

# Affine structure from motion

- Affine projection is a linear mapping + translation in inhomogeneous coordinates




$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} = \mathbf{AX} + \mathbf{t}$$

Projection of world origin

1. We are given corresponding 2D points (**x**) in several frames
2. We want to estimate the 3D points (**X**) and the affine parameters of each camera (**A**)

# Step 1: Simplify by getting rid of **t**: shift to centroid of points for each camera

$$\mathbf{x}_i = \mathbf{A}_i \mathbf{X} + \mathbf{t}_i \qquad \hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}_{ik}$$

$$\mathbf{x}_{ij} - \frac{1}{n}\sum_{k=1}^{n}\mathbf{x}_{ik} = \mathbf{A}_i\mathbf{X}_j + \mathbf{t}_i - \frac{1}{n}\sum_{k=1}^{n}\left(\mathbf{A}_i\mathbf{X}_k + \mathbf{t}_i\right) = \mathbf{A}_i\left(\mathbf{X}_j - \frac{1}{n}\sum_{k=1}^{n}\mathbf{X}_k\right) = \mathbf{A}_i\hat{\mathbf{X}}_j$$

$$\hat{\mathbf{x}}_{ij} = \mathbf{A}_i \hat{\mathbf{X}}_j$$

2d normalized point (observed)

Linear (affine) mapping

3d normalized point

Suppose we know 3D points and affine camera parameters …

then, we can compute the observed 2d positions of each point

$$\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_n \end{bmatrix}$$

3D Points (3xn)

Camera Parameters (2mx3)

What if we instead observe corresponding 2d image points?

Can we recover the camera parameters and 3d points?

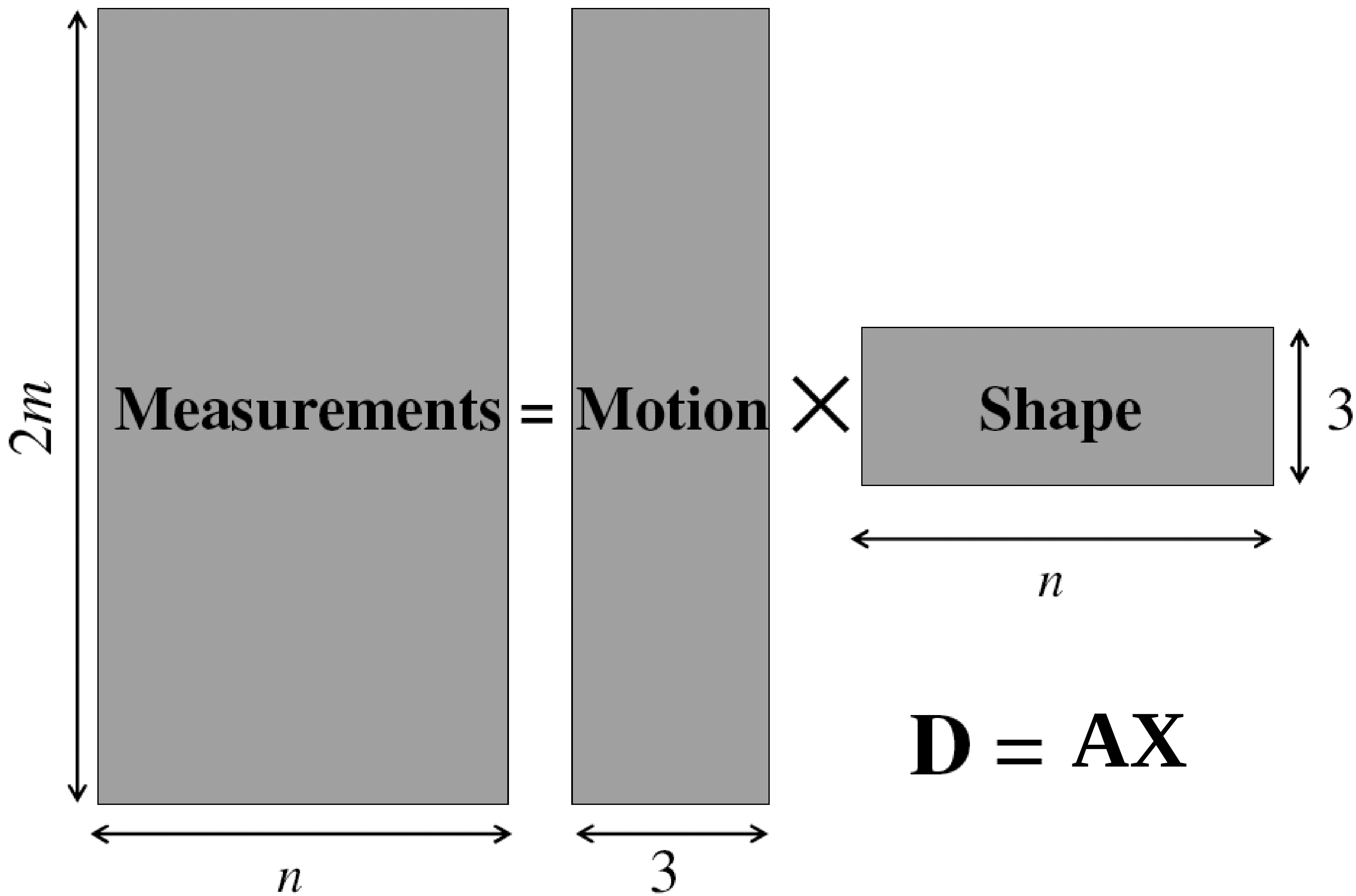
cameras ($2m$)

$$\mathbf{D} = \begin{bmatrix} \hat{\mathbf{x}}_{11} & \hat{\mathbf{x}}_{12} & \cdots & \hat{\mathbf{x}}_{1n} \\ \hat{\mathbf{x}}_{21} & \hat{\mathbf{x}}_{22} & \cdots & \hat{\mathbf{x}}_{2n} \\ & & \ddots & \\ \hat{\mathbf{x}}_{m1} & \hat{\mathbf{x}}_{m2} & \cdots & \hat{\mathbf{x}}_{mn} \end{bmatrix} \overset{?}{\Rightarrow} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_n \end{bmatrix}$$

points ($n$)

What rank is the matrix of 2D points?

# Factorizing the measurement matrix

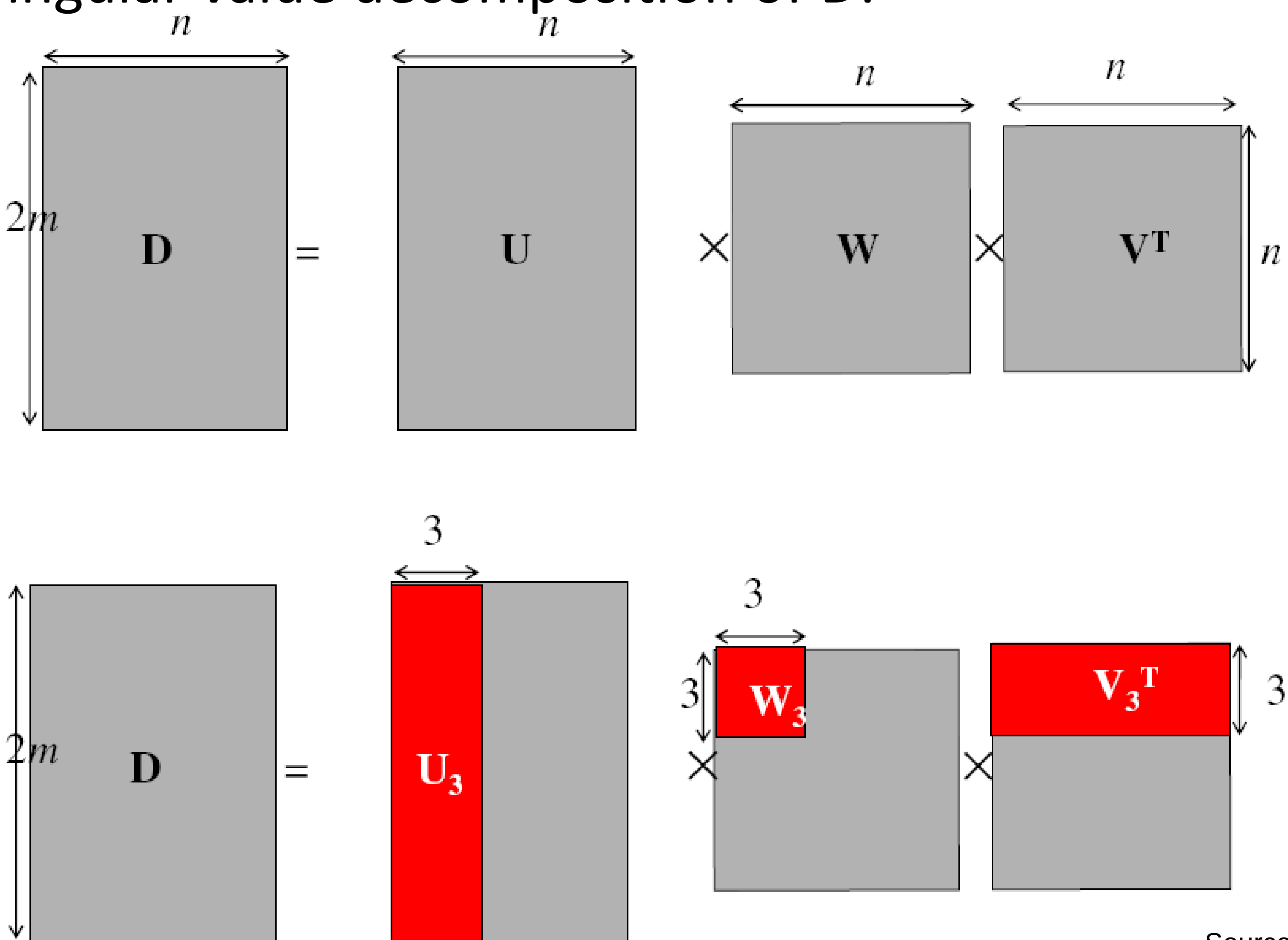



$$\mathbf{D} = \mathbf{A}\mathbf{X}$$

Source: M. Hebert

# Factorizing the measurement matrix

- Singular value decomposition of D:




Source: M. Hebert

# Factorizing the measurement matrix

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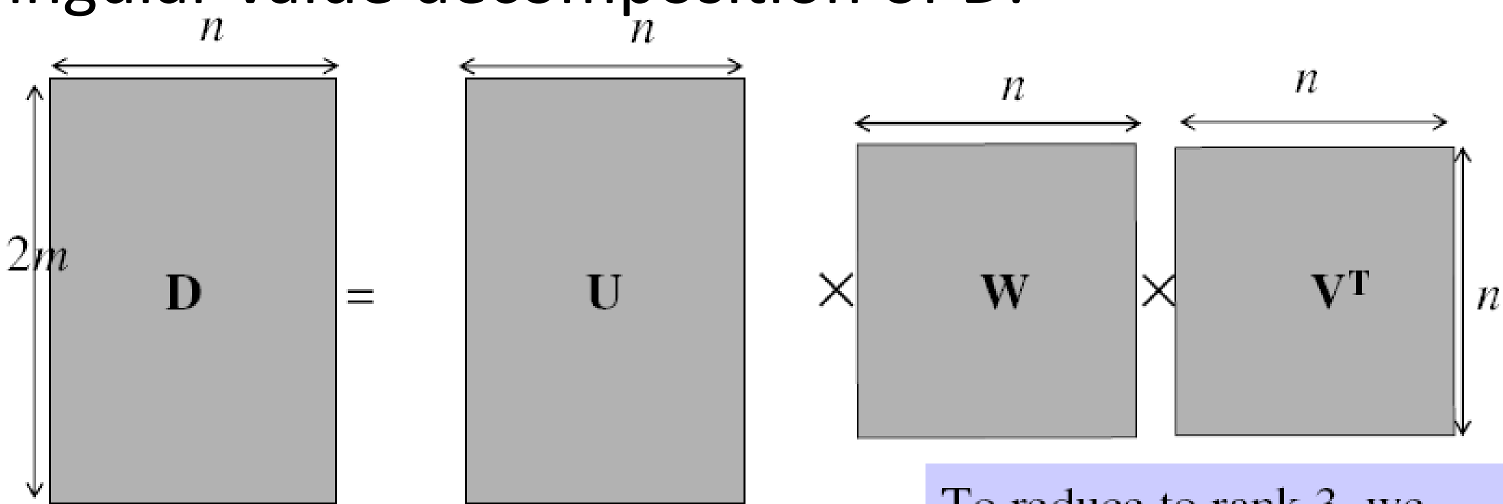
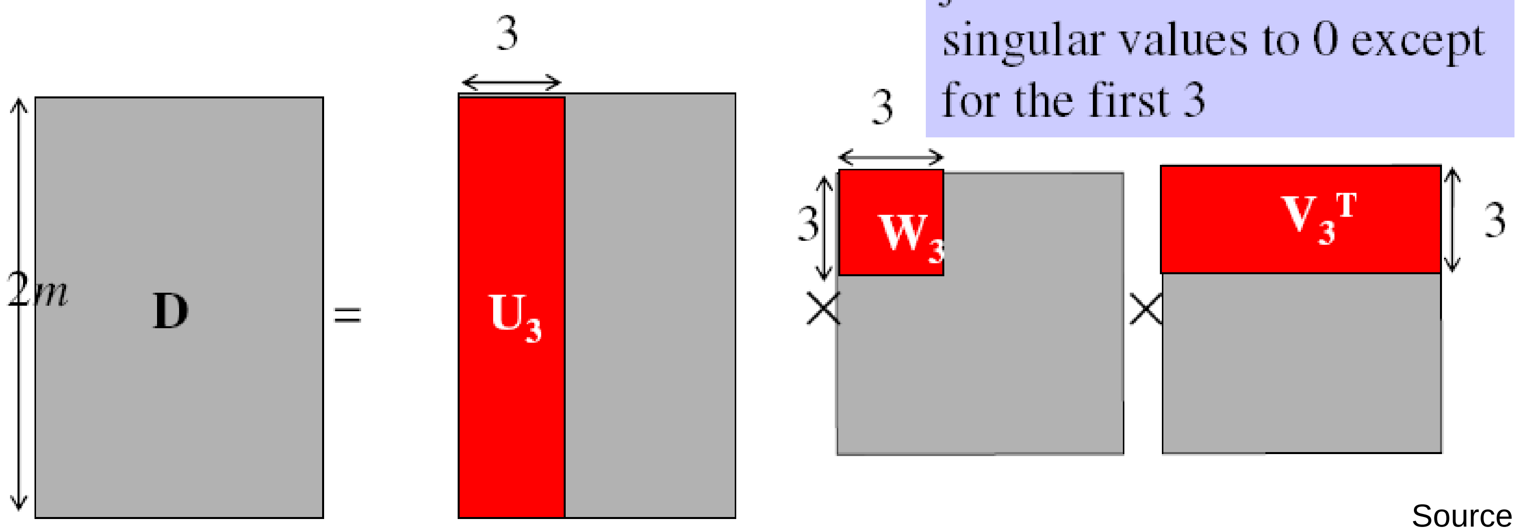


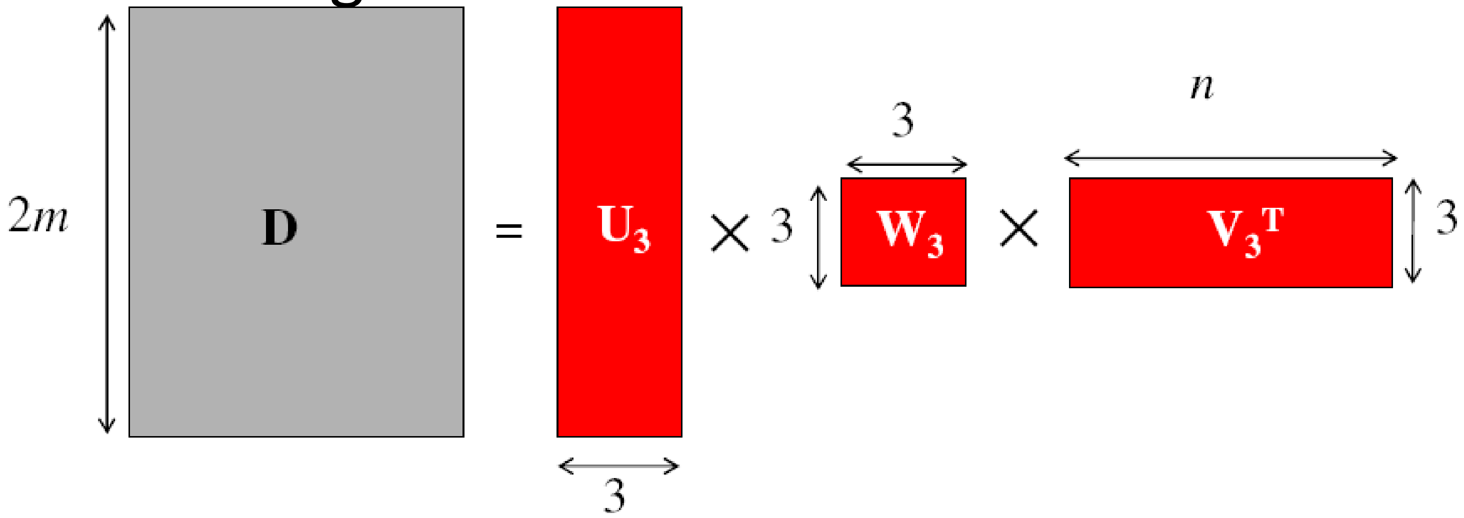

To reduce to rank 3, we just need to set all the singular values to 0 except for the first 3




Source: M. Hebert

# Factorizing the measurement matrix

- Obtaining a factorization from SVD:




Source: M. Hebert

# Factorizing the measurement matrix

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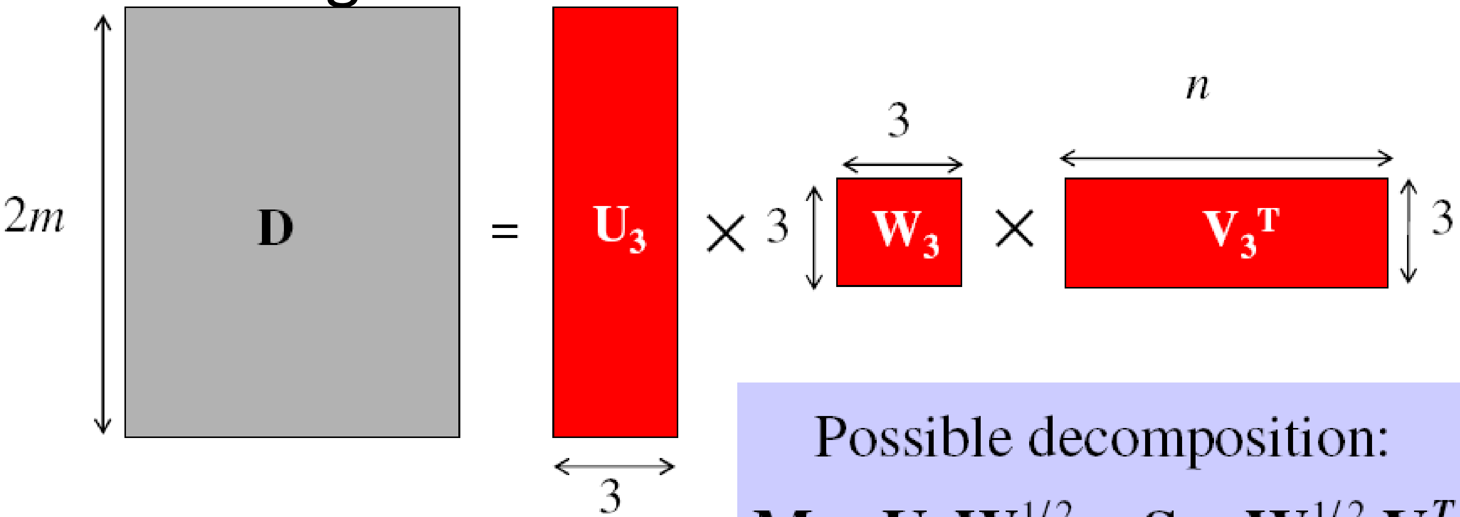
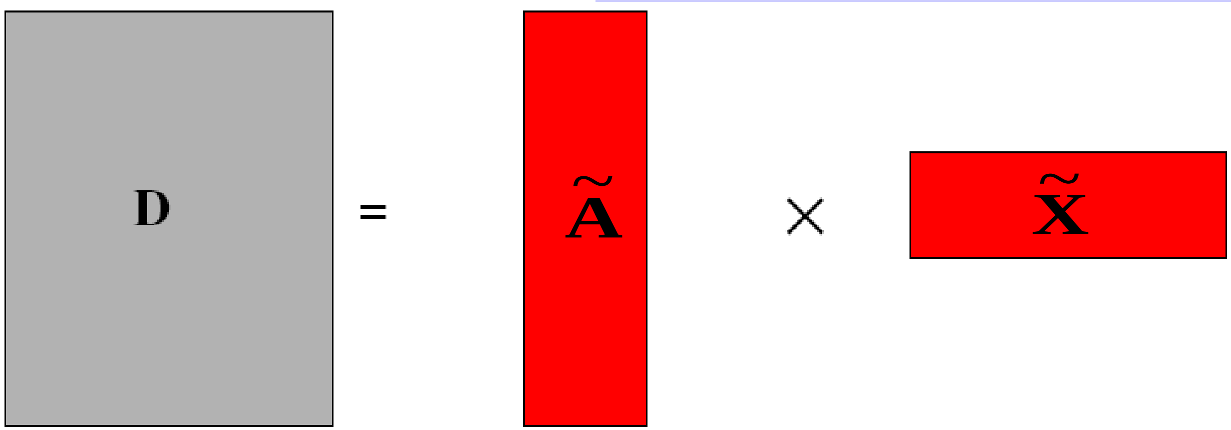


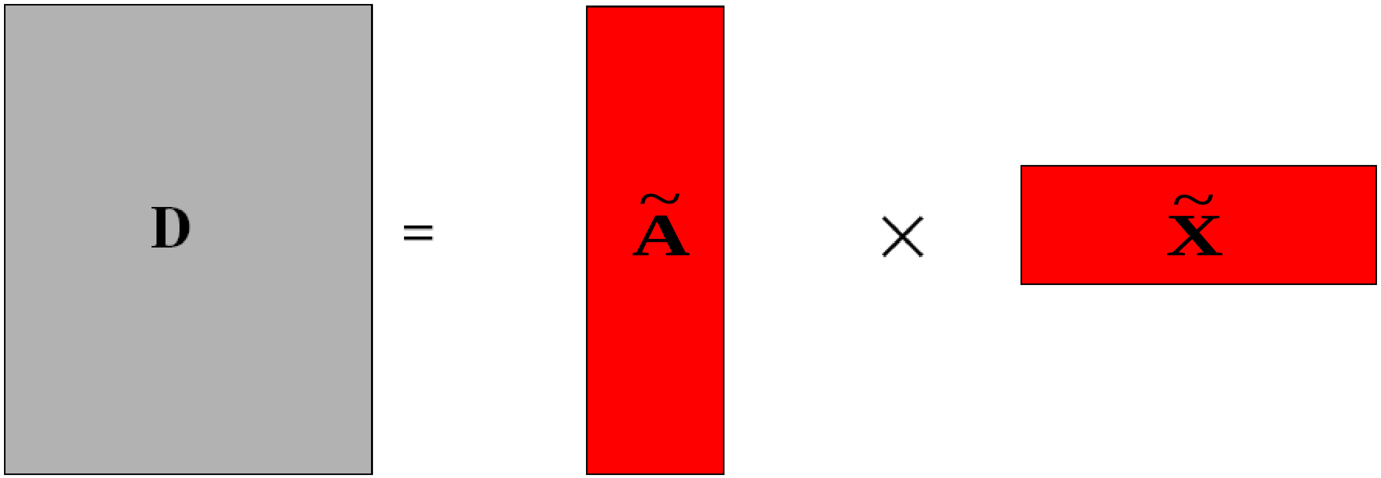

Possible decomposition:

$$\mathbf{M} = \mathbf{U}_3 \mathbf{W}_3^{1/2} \qquad \mathbf{S} = \mathbf{W}_3^{1/2}\, \mathbf{V}_3^T$$




Source: M. Hebert

# Affine ambiguity

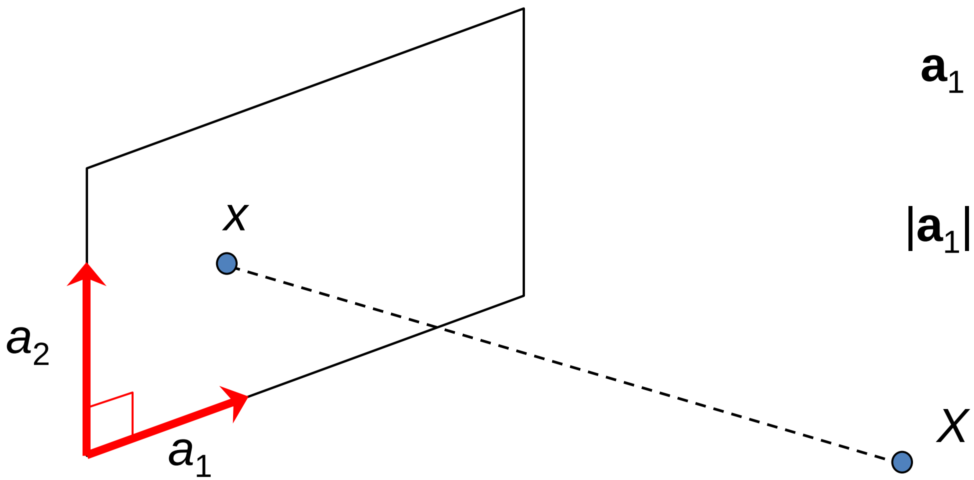



- The decomposition is not unique. We get the same **D** by using any 3×3 matrix **C** and applying the transformations $\mathbf{A} \rightarrow \mathbf{AC}$, $\mathbf{X} \rightarrow \mathbf{C}^{-1}\mathbf{X}$
- That is because we have only an affine transformation and we have not enforced any Euclidean constraints (like forcing the image axes to be perpendicular, for example)

Source: M. Hebert

# Eliminating the affine ambiguity

- Orthographic: image axes are perpendicular and of unit length




$\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$

$|\mathbf{a}_1|^2 = |\mathbf{a}_2|^2 = 1$

Source: M. Hebert

# Solve for orthographic constraints

Three equations for each image i

$$\tilde{\mathbf{a}}_{i1}^T \mathbf{C}\mathbf{C}^T \tilde{\mathbf{a}}_{i1} = 1$$
$$\tilde{\mathbf{a}}_{i2}^T \mathbf{C}\mathbf{C}^T \tilde{\mathbf{a}}_{i2} = 1 \quad \text{where} \quad \tilde{\mathbf{A}}_i = \begin{bmatrix} \tilde{\mathbf{a}}_{i1}^T \\ \tilde{\mathbf{a}}_{i2}^T \end{bmatrix}$$
$$\tilde{\mathbf{a}}_{i1}^T \mathbf{C}\mathbf{C}^T \tilde{\mathbf{a}}_{i2} = 0$$

- Solve for **L = CC**$^{\mathbf{T}}$
- Recover **C** from **L** by Cholesky decomposition: **L = CC**$^{\mathbf{T}}$
- Update **A** and **X**: **A = ÃC, X = C**$^{\mathbf{-1}}$**X̃**

# Algorithm summary

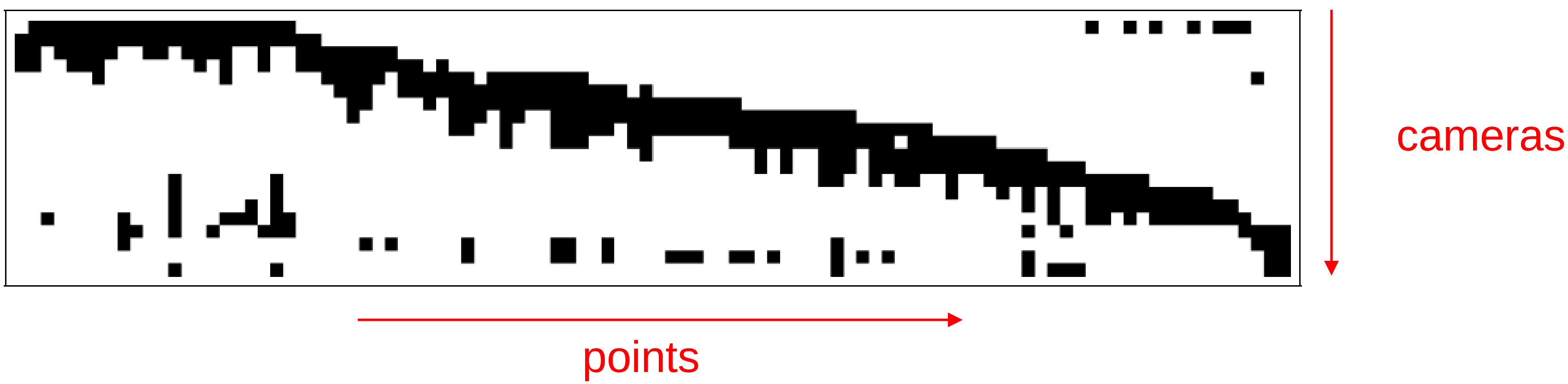
- Given: *m* images and *n* tracked features $\mathbf{x}_{ij}$
- For each image *i, c*enter the feature coordinates
- Construct a $2m \times n$ measurement matrix **D**:
  - Column *j* contains the projection of point *j* in all views
  - Row *i* contains one coordinate of the projections of all the *n* points in image *i*
- Factorize **D**:
  - Compute SVD: $\mathbf{D} = \mathbf{U}\,\mathbf{W}\,\mathbf{V}^T$
  - Create $\mathbf{U}_3$ by taking the first 3 columns of **U**
  - Create $\mathbf{V}_3$ by taking the first 3 columns of **V**
  - Create $\mathbf{W}_3$ by taking the upper left $3 \times 3$ block of **W**
- Create the motion (affine) and shape (3D) matrices:

  $\mathbf{A} = \mathbf{U}_3\mathbf{W}_3^{½}$ and $\mathbf{X} = \mathbf{W}_3^{½}\,\mathbf{V}_3^T$
- Eliminate affine ambiguity

Source: M. Hebert

# Dealing with missing data

- So far, we have assumed that all points are visible in all views
- In reality, the measurement matrix typically looks something like this:



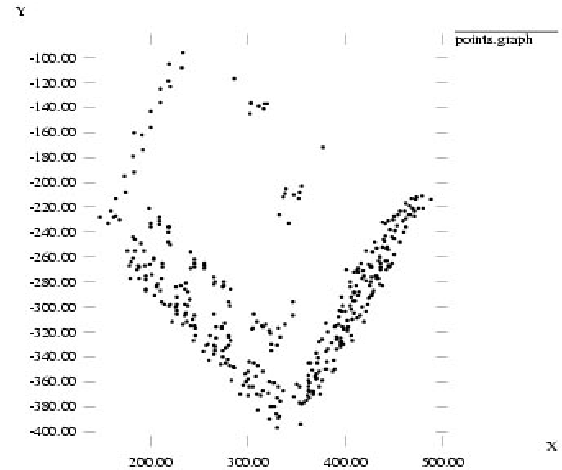

One solution:

- solve using a dense submatrix of visible points
- Iteratively add new cameras

# Reconstruction results (your HW 3.4)

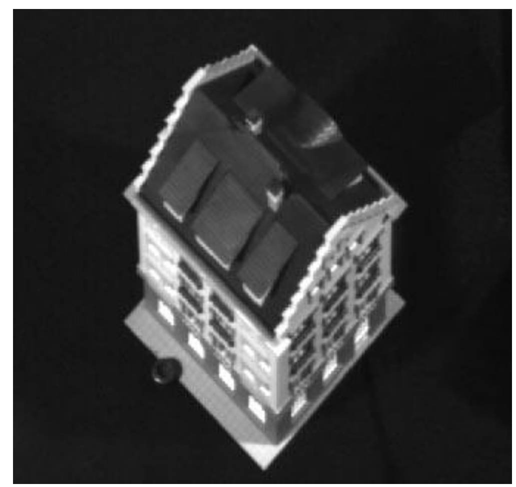
1

60




120

150



C. Tomasi and T. Kanade. [Shape and motion from image streams under orthography: A factorization method.] *IJCV*, 9(2):137-154, November 1992.

# Further reading

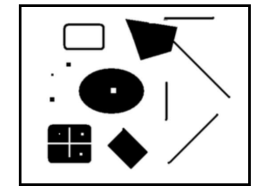
- Short explanation of Affine SfM: class notes from Lischinksi and Gruber

  http://www.cs.huji.ac.il/~csip/sfm.pdf

- Clear explanation of epipolar geometry and projective SfM
  - http://mi.eng.cam.ac.uk/~cipolla/publications/contributionToEditedBook/2008-SFM-chapters.pdf

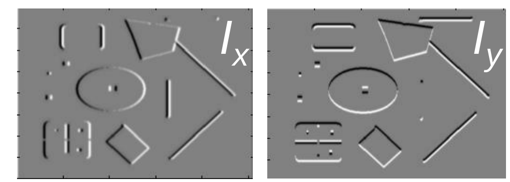
# Review of Affine SfM from Interest Points

## 1. Detect interest points (e.g., Harris)

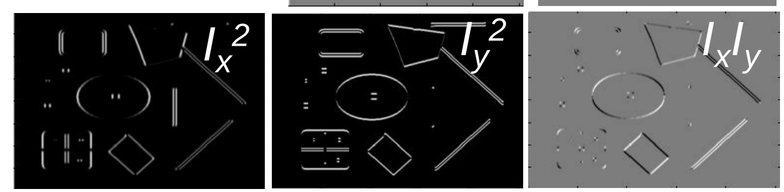


$$\mu(\sigma_I, \sigma_D) = g(\sigma_I) * \begin{bmatrix} I_x^2(\sigma_D) & I_x I_y(\sigma_D) \\ I_x I_y(\sigma_D) & I_y^2(\sigma_D) \end{bmatrix}$$

1. Image derivatives

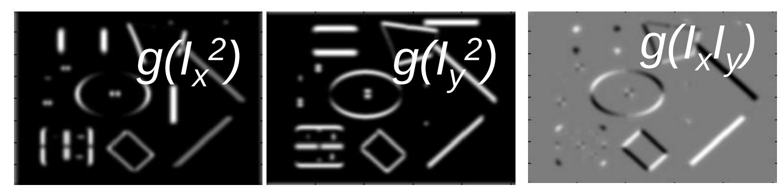



2. Square of derivatives




$$\det M = \lambda_1 \lambda_2$$
$$\operatorname{trace} M = \lambda_1 + \lambda_2$$

3. Gaussian filter $g(\sigma_I)$




4. Cornerness function – both eigenvalues are strong

$$har = \det[\mu(\sigma_I, \sigma_D)] - \alpha[\operatorname{trace}(\mu(\sigma_I, \sigma_D))^2] =$$
$$g(I_x^2)g(I_y^2) - [g(I_x I_y)]^2 - \alpha[g(I_x^2) + g(I_y^2)]^2$$

5. Non-maxima suppression

# Review of Affine SfM from Interest Points

## 2. Correspondence via Lucas-Kanade tracking

a) Initialize (x',y') = (x,y)

b) Compute (u,v) by

Original (x,y) position

$I_t = I(x', y', t+1) - I(x, y, t)$

$$\begin{bmatrix} \sum I_x I_x & \sum I_x I_y \\ \sum I_x I_y & \sum I_y I_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} \sum I_x I_t \\ \sum I_y I_t \end{bmatrix}$$

2nd moment matrix for feature patch in first image

displacement

c) Shift window by (u, v): `x'=x'+u;  y'=y'+v;`

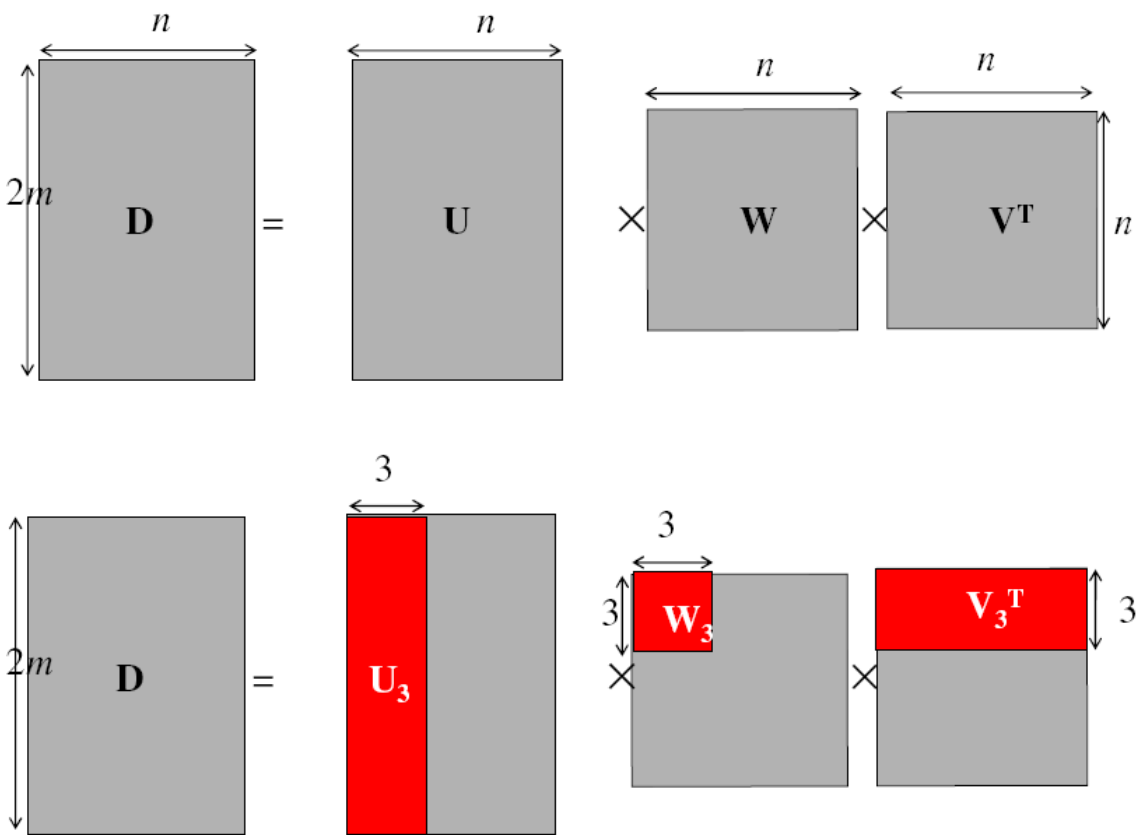
d) Recalculate $I_t$

e) Repeat steps 2-4 until small change

- Use interpolation for subpixel values

# Review of Affine SfM from Interest Points

3. Get Affine camera matrix and 3D points using Tomasi-Kanade factorization

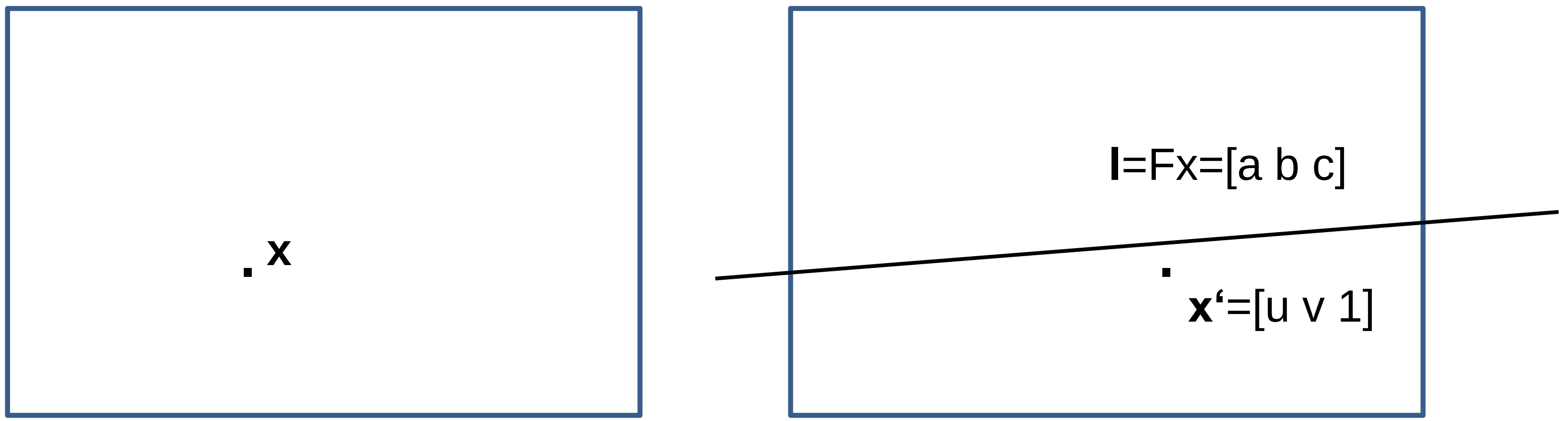



+ Solve for orthographic constraints

# Tips for HW 3

- Problem 1: vanishing points
  - Use lots of lines to estimate vanishing points
  - For estimation of VP from lots of lines, see single-view geometry chapter, or use robust estimator of a central intersection point
  - For obtaining intrinsic camera matrix, numerical solver (e.g., `fsolve` in matlab) may be helpful
- Problem 3: epipolar geometry
  - Use reprojection distance for inlier check (make sure to compute line to point distance correctly)
- Problem 4: structure from motion
  - Use Matlab's `chol` and `svd`
  - If you weren't able to get tracking to work from HW2 can use provided points

# Distance of point to epipolar line




$$d(l, x') = \frac{|au + bv + c|}{\sqrt{a^2 + b^2}}$$

# Next class

- Clustering and using clustered interest points for matching images in a large database