1. Executability Conditions for Rewrite

Theories

Given a rewrite theory \( R = (\Sigma, \delta, EUB, R) \), the problem we may have is that the relation \( \rightarrow_{R/EUB} \) that faithfully describes \( \text{one-step transitions in} \) \( J_R \), in the sense that we have the equivalence

\[
[u] \rightarrow [v] \iff \exists [u] \rightarrow^{R/EUB} [v]
\]

may be \underline{decidable}: we may not be able to know whether a single step from \( u \) is possible with \( \rightarrow_{R/EUB} \).

This is obviously a bad situation. But it is quite understandable. The problem is that we have not yet identified executability conditions for \( R \), similar to those for a \underline{more functional module} \( \text{func}(\Sigma, EUB) \) \underline{end} where we require the relation \( \rightarrow_{E/B} \) to be \underline{ground confluent}:

\[
\begin{align*}
\overset{*}{A} & \rightarrow_{E/B} u \\
\overset{*}{E} \rightarrow_{E/B} w &= \overset{*}{E} \rightarrow_{E/B} w\\
\end{align*}
\]

and \underline{terminating}. \[\]
So, the question is: are there similar executability conditions that we should require for a Mandle system's module of the form: $\text{mod}(\Sigma, \phi, EUB, R)$ and $\text{mod}(\Sigma, EUB)$?

Obviously, since the equational part $(\Sigma, EUB)$ is just the specification of the module's auxiliary functions to perform functional computations in a state [the module's static], we should view this part as its functional sub-module, and require the same executability conditions as if we had explicitly specified as such: $\text{mod}(\Sigma, EUB)$ or $\text{mod}(\Sigma, \phi)$, and then implemented it. That is, $\Rightarrow_{\text{EUB}}$ should be [ground] confluent and terminating.

But what about $R$? We know that $\Rightarrow_{R^B/EUB}$ is too complicated. Can we simulate by some other relation that makes it decidable whether we can perform a one-step transition under suitable conditions?

A key insight comes from realizing that, thanks to the convergence [confluence and termination] of $\Rightarrow_{EUB}$ modulo $B$, because, up to $B$-equality, an $EUB$-equivalence class $[u]_{EUB}$ can be...
uniquely represented/summarized by its normal form $u!_{E/B}$.

But computing the $E/B$-normal form of a term $u$, say, $v$, is a relation that can be denoted:

$$u \xrightarrow{E/B!} v,$$

and the re-composed relation:

$$u \xrightarrow{E/B!} v \overset{R^0/B}{\rightarrow} w$$

has two good properties:

1. \( \xrightarrow{E/B!} ; \overset{R^0/B}{\rightarrow} \), abbreviated to: \( \xrightarrow{E/B! R^0/B} \)

   is decidable; i.e., given two terms $u$, and $w$, we can decide in a finite number of steps whether

   \( \exists w' \in [w]_B \) such that \( u \overset{E/B! R^0/B}{\rightarrow} w' \)

   assuming the set $R$ of rules is finite, and $B$

   and $E$ of equations are

   in any combination of $A, C$, and

2. We obviously have that a containment of relations:

   \( \xrightarrow{E/B! R^0/B} C \overset{E/B! R^0/B^*}{\rightarrow} \)

   \( \xrightarrow{E/B! R^0/EUB} \)

   But what we, obviously would like to have is the equivalence:
\[ [u] \xrightarrow{\mathcal{E}/\mathcal{B}!R^\Phi/B} [v] \quad \Leftrightarrow \quad [u] \xrightarrow{\mathcal{E}/\mathcal{B}!R^\Phi/B} [v] \quad \Leftrightarrow \quad [u] \xrightarrow{E/B!R^\Phi/B} [v] \]

To obtain such an equivalence it is a good idea to see clearly what we want, and then find a requirement for it. Notice that, pictorially, we can view successive applications of \(E/B!R^\Phi/B\) as a \textit{stairway descent process}:

\[ U_0 \xrightarrow{E/B!R^\Phi/B} U_1 \xrightarrow{E/B!R^\Phi/B} U_2 \xrightarrow{\cdots} \]

which is exactly the way the Mende interpreter computes with a system module. Obviously, we always have the implication:

\[ [u] \xrightarrow{E/B!R^\Phi/B} [v] \Rightarrow [u] \xrightarrow{R} [v] \]
The burning question is the completeness question: are we missing something with $\frac{E \oplus R \oplus B}{E \oplus B}$? That is, are we missing reachable states? Do we really also have an implication the other way?, i.e.,

$$[u] \xrightarrow{E \oplus B \oplus R \oplus B} [v] \iff [u] \xrightarrow{R} [v]$$

In general we may not. Suppose $\Sigma$ has just three constants, $a$, $b$, $c$, $E = \{ a = b \}$, which is confluent and terminating!, and $R = \{ a \rightarrow c \}$.

We of course have $[a] \xrightarrow{R} [c]$, but we do not have $[a] \xrightarrow{E \oplus B \oplus R \oplus B} [c]$. In fact, $\xrightarrow{E \oplus b}$ is the empty relation!

This failure of completeness looks like this:

$\begin{array}{c}
a \\
\downarrow E!
\end{array} \xrightarrow{R} \begin{array}{c}
b \\
\downarrow ??
\end{array} \rightarrow c$

So we just should require that $b$ should simulate $a$. 
This is the so-called coherence property:

\[ A \xrightarrow{R^g/B} W \]
\[ E/B \]
\[ W' \xrightarrow{E/B} W'!E/B \]
\[ B \]

Or, put even more compactly, but equivalently:

\[ A \xrightarrow{R^g/B} W \]
\[ E/B \]
\[ W' \xrightarrow{E/B} W'!E/B \]
\[ E/B \]

We call the property ground coherence if we only require this for ground terms, which is quite enough for our purposes, since \( T \) is a model where the states are concrete states, i.e., of the form \([u]_{EUB}\) with \( u \in T \) a ground term.

And, of course, what we want to prove is:
Main Theorem. If \( R = (\Sigma, \phi, EUB, R) \) is ground coherent, then we have the equivalence:

\[
[u] \xrightarrow{R} [v] \iff [u] \xrightarrow{E/B!R^0/B} [v].
\]

Proof. We only need to prove the \((\Rightarrow)\) part. But this follows from the following diagram:

\[
\begin{array}{c}
\exists u, v' \\
\end{array}
\]

That is, we have \( U \xrightarrow{E/B!R^0/B} v_0 \) with \( v_0 = v' \)

and therefore, \( [u] \xrightarrow{E/B!R^0/B} [v'] \), as desired! \( \square \)
The big picture! What have we accomplished?

A no mean feat! Namely, to turn the beast of the generally hopeless and undecidable rewrite modules EUB relation $\text{R}^\circ$ by a much simpler relation $\text{EUB}! \text{R}^\circ/\text{B}$ under the assumptions [grand]

that $\text{E}/\text{B}$ is convergent and $\text{R}^\circ$ is coherent with $\text{E}$ modulo $\text{B}$.

In other words, what the relation $\text{EUB}! \text{R}^\circ/\text{B}$ efficiently implements is rewriting modulo EUB.

A very practical question: Assume that $\text{E}$ is already convergent modulo $\text{B}$. How can we:

1. Check that $\text{R}^\circ$ is coherent with $\text{E}$ modulo $\text{B}$?

2. Make $\text{R}^\circ$ so in case the check fails?

The answer to 1 is provided by Mandle's
Church-Rosser Checker and Coherence Checker tool, whose theoretical foundations are explained in the Durán-Meseguer paper available in the course web page.

A recent answer to (2) has been given by defining an automatable method of coherence completion that makes a rewrite theory \( R \) coherent by a transformation process \( R \rightarrow \hat{R} \), where \( \hat{R} \) is ground coherent and semantically equivalent to \( R \), i.e.

\[
[u] \overset{R}{\rightarrow} [v] \iff [u] \overset{\hat{R}}{\rightarrow} [v]
\]

This is also described in a paper by Meseguer on coherence completion also available in the course web page.

An even more concrete model of \( \hat{R} \)

The category \( \hat{R} \) is of course a good mathematical model of the concurrent computations of the system specified by \( R \). But it does not take account of any executability requirements.
But if $\mathcal{R} = (\Sigma, \varphi, E \cup B, R)$ has all the good properties of (1) ground convergence of $E$ modulo $B$, and (2) ground coherence of $R^0$ with $E$ modulo $B$, a much more concrete model of $\mathcal{R}$ is possible, namely the one implicitly used by Mande. In this model states are $B$-equivalence classes of terms in normal form, i.e., $[u!_E B]_B$ and transitions between such states, denoted $[u]_B \rightarrow [v]_B$ hold iff $U \xrightarrow{R^0/B} W \xrightarrow{E/B!} W_0 = V$. That is,

$$[u]_B \rightarrow [v]_B \iff [u]_B \xrightarrow{R^0/B; E/B!} [v]_B$$

This completes the picture in pg. 4 as follows:

This is, for example, the way Mande's search command works: what Mande stores are the reached states of the form $[u!_E B]_B$