1. More on Sequential Systems: Topmost Theory

Recall that a rewrite theory $R = (\Sigma, \phi, EUB, R)$ specifies a sequential system iff it has: (i) no sideways parallelism, and (ii) no nested concurrency.

A very useful class of sequential rewrite theories is provided by so-called topmost rewrite theories where any one-step rewrite must happen at the top of the term, i.e., must be of the form:

$$[u] \xrightarrow{[v]} \frac{l(v_1, \ldots, v_n)}{t(v_1, \ldots, v_n) \in [u], t'(v_1, \ldots, v_n) \in [v]}.$$

with $t : t \to t' \in R$, $t(v_1, \ldots, v_n) \in [u]$, $t'(v_1, \ldots, v_n) \in [v]$.

Definition Let $R = (\Sigma, \phi, EU B, R)$ be a rewrite theory with $\Sigma = ((S, \leq), \Sigma)$ an ordered sorted signature.

Then $R$ is called topmost iff there is a connected component of the graph/point of sorts $[s] \in S/\leq \times \Sigma^{+}$ such that:

1. any $l : t \to t'$ in $R$ has $l[s(t), l[s(t')] \subseteq [s]$ nonempty

2. For any term $u \in T_{\Sigma}(X)$, and position $p \in pos(u) \setminus \{\varepsilon\}$, if $l[s(u|_p)] \subseteq [s]$, then $p$ is a frozen position in $u$ according to $\phi$. 


An easy sufficient condition for $R$ to be topmost on hand $[3]$, which does not depend on \( \Phi \), but only on $\Sigma$ is:

- for any $f: s_1, \ldots, s_n \rightarrow s'$ in $\Sigma$, $m > 1$, $s_1, \ldots, s_n \notin [3]$


The following lemma is left as an exercise:

**Lemma.** If $R = (\Sigma, \Phi, EUB, R)$ is topmost, then $R$ is sequential.

Topmost rewrite theories look very restrictive. However, it is very often quite easy to transform a rewrite theory $R$ into a topmost one $R^t$ such that $R$ and $R^t$ have the same interleaving computations. That is, in $R^t$ we lose all concurrency, but for some reasoning purposes, e.g., LTL model checking, or reasoning about invariants, maybe the only things that matter is whether we can reach a given state. And from that point of view $R$ and $R^t$ become reachability equivalent.

Let us illustrate the $R \rightarrow R^t$ transformation.
for the case of concurrent object-oriented systems, where, remember that all rules have the [very] general form:

\[
\begin{align*}
\forall \mathcal{L} & : <O_1|\text{ATTS}_1> ... <O_n|\text{ATTS}_n> \mathcal{M}_1 ... \mathcal{M}_n \\
\rightarrow & <O_{i_1}|\text{ATTS}_{i_1}'> ... <O_{i_q}|\text{ATTS}_{i_q}'> <Q_1|\text{ATTS}_1''> ... <Q_k|\text{ATTS}_k''> \mathcal{M}_1' ... \mathcal{M}_q'
\end{align*}
\]

We can make such \( \mathcal{R} \) concurrent by:

1. Adding a new sort \( \text{System} \) not belonging to any connected component of sorts, and
2. Adding an operator \( [-?] : \text{Configuration} \rightarrow \text{System} \)
3. Transforming any rule of the form \((\ast)\) into a rule:

\[
\begin{align*}
\forall \mathcal{L} & : \{<O_1|\text{ATTS}_1> ... <O_n|\text{ATTS}_n> \mathcal{M}_1 ... \mathcal{M}_n \} \\
\rightarrow & \{<O_{i_1}|\text{ATTS}_{i_1}'> ... <O_{i_q}|\text{ATTS}_{i_q}'> <Q_1|\text{ATTS}_1''> ... <Q_k|\text{ATTS}_k''> \mathcal{M}_1' ... \mathcal{M}_q' \}
\end{align*}
\]

where \( \mathcal{C} \) is a fresh new variable of sort \( \text{Configuration} \).

Lemma. Prove that for \( \mathcal{R} \) the rewrite theory of a concurrent object system, the above transformation \( \mathcal{R} \rightarrow \mathcal{R}^+ \) is such that:

1. \( \mathcal{R}^+ \) is topmost
2. \( \mathcal{R} \) and \( \mathcal{R}^+ \) enjoy the property that their interleaving computations are in a bijective correspondence.
2. The Big Picture

At this point in the course, we can take a bird's eye view of the entire field of concurrency. This view need not be exhaustive, but can help us get the lay of the land; furthermore, we can take a peek at an area we did not have a chance to visit, yet quite important, namely, real-time concurrent systems. Here is the big picture:
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3. The \( \rightarrow_r \) and \( \rightarrow_{R/EuB} \) Rewrite Relations

Thanks to lecture 22's Sequentialization Lemma, we know that any concurrent computation \([u] \xrightarrow{[x]} [v]\) in a rewrite theory \(R\) has [possibly many] equivalent interleaving descriptions rooted \([x] = [\beta_1, \ldots, \beta_n]\), with the \(\beta_i\): one-step rewrites.

Let us adopt the notational convention:

\[
[u] \rightarrow [v] \quad \text{iff} \quad [u] \xrightarrow{[t[l(v_{i_1}, v_{i_2})p]]} [v] \quad \text{for some}
\]

That is, iff \([u] \rightarrow [v]\) describes the existence of a one-step rewrite in \(R\).

Of course, the states of the system described by \(R\) are \(EuB\)-equivalence classes \([u]_{EuB}\). But this is a quite abstract description, since: 1) such classes are in general infinite, and 2) the equivalence relation \(=_{EuB}\) is in general undecidable.

Therefore, it is useful to define a relation \(\rightarrow_{R^0/EuB}\) at the level of terms, that is, a binary relation
\[ \frac{R^\theta/EUB}{\Rightarrow} \subseteq T_\Sigma \times T_\Sigma \]  

or, more generally,

\[ \frac{R^\theta/EUB}{\Rightarrow} \subseteq T_\Sigma(X) \times T_\Sigma(X) \]

such that given

\[ R = (\Sigma, \phi, EUB, R) \] we have:

\[ [u] \rightarrow [v] \] iff \( \exists v \in [v] \) s.t. \( [u] \rightarrow [u'] \land v' \in [v] \)

Of course, the relation \( \frac{R^\theta/EUB}{\Rightarrow} \) defines also a relation on \( EUB \)-equivalence classes, namely:

\[ [t] \rightarrow [t'] \]. From this way, all boils down to:

\[ [u] \rightarrow [v] \] iff \( [u] \rightarrow [v] \) \( \iff [u] \rightarrow [v] \)

That is, on equivalence classes both relations coincide.

Before defining \( \frac{R^\theta/EUB}{\Rightarrow} \) we can define a much simpler and decidable relation, namely, the syntactic rewriting relation

\[ \frac{R^\theta}{\Rightarrow} \subseteq T_\Sigma \times T_\Sigma \]

which is defined as follows:

\[ U \rightarrow R^\theta \ U \iff \begin{cases} \exists \theta, \text{not } \phi\text{-fresh} \text{ position in } U \text{ s.t. } \exists \theta \text{ substitution } \theta \in \theta \text{ in } R \end{cases} \]

\[ U = U[t\theta]_p \]

\[ U = U[t'\theta]_p \]
Pictorially:

we can now easily define \( R^\phi/EUB \) as a composition of relations:

\[
\overrightarrow{R^\phi/EUB} \quad \overset{\text{def}}{=} \quad \overrightarrow{EUB}; \quad \overrightarrow{R^\phi}
\]

In other words:

\[
U \xrightarrow{R^\phi/EUB} U' \iff \exists u' \in U \text{ s.t. } U' \xrightarrow{R^\phi} V
\]

Prove as an exercise:

Lemma. For any rewrite theory \( R = (\Sigma, \phi, EUB, R) \)

we have the equivalence:

\[
[U] \xrightarrow{R} [U'] \iff [U] \xrightarrow{R^\phi/EUB} [U]
\]

3. A Serious Executability Problem

In general the relation \( =_{EUB} \) is undecidable, i.e., semi-decidable.

This actually means that given a term \( u \in T_\Sigma \) we may not able to decide whether a one-step rewrite
$u \xrightarrow{R^0/EVB} v$ is even possible for some $v$. The problem is that we have to find a $u' \in [u]$ such that $u' \xrightarrow{R^0} v$ for some $v$, but such a $u'$ may not exist, and we may never find out whether it does exist in case it doesn't.

A much better situation arises for axioms $B$ such as any combination of associativity and/or commutativity and/or identity axioms for which the problem:

$$\exists u' \in [u] \text{ s.t. } u' \xrightarrow{R^0} v \text{ is decidable.}$$

Since, in particular, such decidability property exists for $B$ any combination of $\text{assoc}, \text{comm}$ and $\text{id}$: attributes, our next order of business will be to reduce the problem $u \xrightarrow{R^0/EVB} v$ to the much simpler problem $u \xrightarrow{R^0/B} v'$ under suitable conditions such that $[v]_{EVB} = [v']_{EVB}$.