Rewrite Theories in General

It is clear that rewriting logic has the underlying equational logic as a parameter: the more general the equational logic, the more general the resulting rewrite theories. For example, we have seen that for order-sorted equational logic rules can have the general form,

\[ l : t \rightarrow t' \iff (\bigwedge_{i} u_i = u'_i) \land (\bigwedge_{j} w_j \rightarrow w'_j). \]

It has also become increasingly clear that frozen operators, that restrict the rewrites allowed below them, are also very useful in practice.
We can illustrate frozen operators with the following nondeterministic choice example (in Maude syntax):

```plaintext
mod CHOICE is
    protecting INT .
    sorts Elt MSet .
    subsorts Elt < MSet .
    ops a b c d e f g : -> Elt .
    op _ _ : MSet MSet -> MSet [assoc comm] .
    op card : MSet -> Int [frozen] .
    eq card(X:Elt) = 1 .
    eq card(X:Elt M:MSet) = 1 + card(M:MSet) .
endm
```
It does not make much sense to rewrite below the cardinality function \texttt{card}, because then the multiset whose cardinality we wish to determine becomes a \textit{moving target}.

If \texttt{card} had not been declared \texttt{frozen}, then the rewrites, \( a \ b \ c \rightarrow b \ c \rightarrow c \) would induce rewrites, \( 3 \rightarrow 2 \rightarrow 1 \), which seems bizarre.

The point is that we think of the kind \texttt{[MSet]} as the \textit{state kind} in this example, whereas \texttt{[Int]} is the \textit{data kind}. By declaring \texttt{card} frozen, we restrict rewrites to the state kind, where they belong.
Rewrite Theories in General (IV)

This leads to the following general definition of a rewrite theory on order-sorted equational logic:

A rewrite theory is a 4-tuple, $\mathcal{R} = (\Sigma, E, \phi, R)$, where:

- $(\Sigma, E)$ is a kind-complete order-sorted equational theory, with, say, kinds $K$, sorts $S$, and operations $\Sigma$
- $\phi : \Sigma \rightarrow \mathcal{P}_{fin}(\mathbb{N})$ is a $K^* \times K$-indexed family of functions assigning to each $f : k_1 \ldots k_n \rightarrow k$ in $\Sigma$ the finite set $\phi(f) \subseteq \{1, \ldots, n\}$ of its frozen argument positions
- $R$ is a set of (universally quantified) labeled conditional rewrite rules of the form (with $t, t'$ and the $w_k, w'_k$ pairs of terms of same kind)

$$l : t \rightarrow t' \iff (\bigwedge_i u_i = u'_i) \land (\bigwedge_j w_j \rightarrow w'_j).$$
Given a rewrite theory $\mathcal{R} = (\Sigma, E, \phi, R)$, and given a $\Sigma$-term $t \in T_\Sigma(X)$, we call a variable $x \in \text{vars}(t)$ frozen in $t$ iff there is a nonvariable position $\alpha \in \mathbb{N}^*$ such that $t/\alpha = f(u_1, \ldots, u_i, \ldots, u_n)$, with $i \in \phi(f)$, and $x \in \text{vars}(u_i)$. Otherwise, we call $x \in X$ unfrozen.

Similarly, given $\Sigma$-terms $t, t' \in T_\Sigma(X)$, we call a variable $x \in X$ unfrozen in $t$ and $t'$ iff it is unfrozen in both $t$ and $t'$. 
Given a rewrite theory $\mathcal{R} = (\Sigma, E, \phi, R)$, the sentences that it proves are universally quantified rewrites of the form, $(\forall X) \ t \longrightarrow t'$, with $t, t' \in T_{\Sigma, E}(X)_k$, for some kind $k$, which are obtained by finite application of the following rules of deduction:

- Reflexivity. For each $t \in T_{\Sigma}(X)$, $\ (\forall X) \ t \longrightarrow t$
• Equality. \((\forall X) u \rightarrow v \quad E \vdash (\forall X) u = u' \quad E \vdash (\forall X) v = v'\)
\[(\forall X) u' \rightarrow v'\]

• Congruence. For each \(f : k_1 \ldots k_n \rightarrow k\) in \(\Sigma\), with \(\{1, \ldots, n\} \rightarrow \phi(f) = \{j_1, \ldots, j_m\}\), with \(t_i \in T_\Sigma(X)_{k_i}, 1 \leq i \leq n\),
and with \(t'_{j_l} \in T_\Sigma(X)_{k_{j_l}}, 1 \leq l \leq m\),
\[(\forall X) t_{j_1} \rightarrow t'_{j_1} \quad \ldots \quad (\forall X) t_{j_m} \rightarrow t'_{j_m}\]
\[(\forall X) f(t_1, \ldots, t_{j_1}, \ldots, t_{j_m}, \ldots, t_n) \rightarrow f(t_1, \ldots, t'_{j_1}, \ldots, t'_{j_m}, \ldots, t_n)\]
• Replacement. For each finite substitution $\theta : X \rightarrow T_{\Sigma}(Y)$, with, say, $X = \{x_1, \ldots, x_n\}$, and $\theta(x_l) = p_l$, $1 \leq l \leq n$, and for each rule in $R$ of the form,

$$l : (\forall X) \, t \rightarrow t' \iff (\bigwedge_i u_i = u'_i) \land (\bigwedge_j w_j \rightarrow w'_j)$$

with $Z = \{x_{j_1}, \ldots, x_{j_m}\}$, the set of unfrozen variables in $t$ and $t'$, then,

$$\left( \bigwedge_r (\forall Y) \, p_{j_r} \rightarrow p'_{j_r} \right)$$

\[
\frac{\left( \bigwedge_i (\forall Y) \, \theta(u_i) = \theta(u'_i) \right) \land \left( \bigwedge_j (\forall Y) \, \theta(w_j) \rightarrow \theta(w'_j) \right)}{(\forall Y) \, \theta(t) \rightarrow \theta'(t')}
\]

where for $x \in X - Z$, $\theta'(x) = \theta(x)$, and for $x_{j_r} \in Z$, $\theta'(x_{j_r}) = p'_{j_r}$, $1 \leq r \leq m$. 
• Transitivity

\[
(\forall X) \, t_1 \rightarrow t_2 \quad (\forall X) \, t_2 \rightarrow t_3 \\
\hline
(\forall X) \, t_1 \rightarrow t_3
\]
Rewriting Logic in Pictures

Reflexivity

\[ t \quad \rightarrow \quad t \]

Equality

\[ u \quad \rightarrow \quad v \]
\[ u' \quad \rightarrow \quad v' \]
Rewriting Logic in Pictures (II)

**Congruence**

\[ f \quad \rightarrow \quad f \]

\[ \cdots \quad \cdots \quad \cdots \quad \cdots \]

**Replacement**

\[ t \quad \rightarrow \quad t' \]

\[ \cdots \quad \cdots \quad \cdots \quad \cdots \]
Transitivity

REW: Logic in Pictures (III)