CS 519: Scientific Visualization

Tensor Visualization

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Some slides adapted Alexandru Telea, *Data Visualization Principles and Practice*

Tensors

1. **What is a tensor?**
   - Describe in terms of principal component analysis
2. **Basic tensor visualization**
   - component visualization
   - anisotropy visualization
   - major eigenvector visualization
3. **Application: Fiber tracking**
   - basic fiber tracking
   - stream tubes
   - hyperstreamlines

What is a tensor?

Explanation 1: Dimensionality

• scalar:	a 0D array of values	e.g. 1 value
• vector:	a 1D array of values	e.g. 3 values
• tensor:	a 2D matrix of values	e.g. 3x3 = 9 values

Explanation 2: Analysis

- scalar: **magnitude** (of some signal at a point in space)
- vector: **magnitude** and **direction** (of some signal at some point in space)
- tensor: **variation** of **magnitude** (of some signal at some point in space)

What is a tensor?

Explanation 3: As a function

- **scalar:** at $\mathbf{x} \in \mathbf{R}^3$, measure some value $s \in \mathbf{R}$
- **vector:** at $\mathbf{x} \in \mathbf{R}^3$, measure some magnitude and direction $\mathbf{v} \in \mathbf{R}^3$
- **tensor:** at $\mathbf{x} \in \mathbf{R}^3$ *and* in a direction $\mathbf{v} \in \mathbf{R}^3$, measure some magnitude $s \in \mathbf{R}$

Fields

So we have different kinds of fields (i.e. **functions** of a variable $\mathbf{x} \in \mathbf{R}^3$):

Scalar fields $s : \mathbf{R}^3 \to \mathbf{R}$

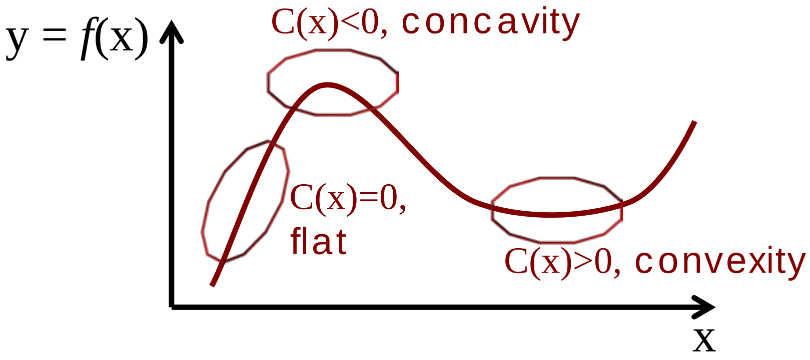
Vector fields $\mathbf{v} : \mathbf{R}^3 \to \mathbf{R}^3$

Tensor fields $\mathbf{T} : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$

Tensor Field Examples

Curvature in 1D

- take a curve $c \subseteq \mathbf{R}^3$
- locally, c can be described as a function $y = f(x)$



- curvature of f $\quad C(x) = \dfrac{\partial^2 f}{\partial x^2}$ $\quad$ (2nd derivative of f)

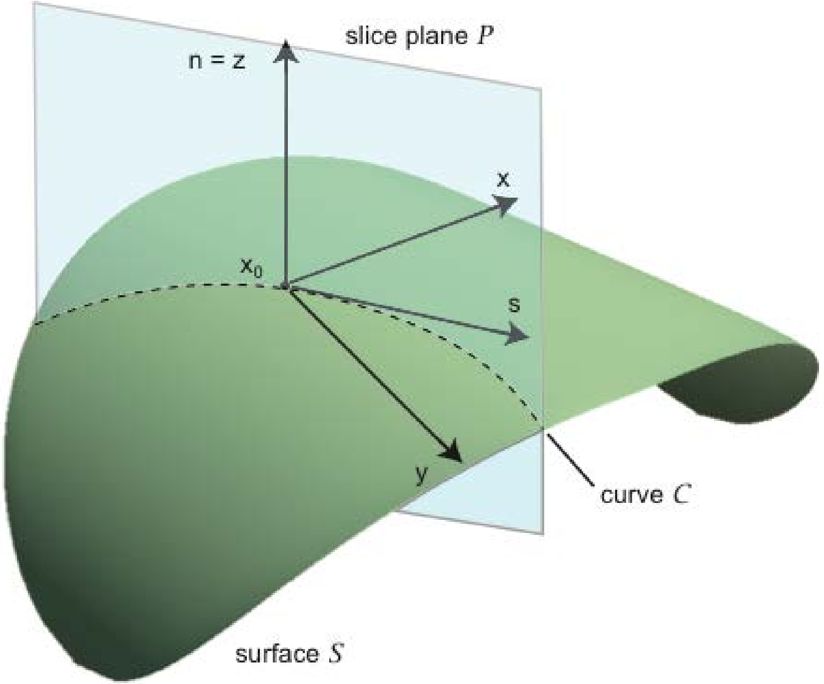
- analytically: $C(x)$ = how quickly the normal $\mathbf{n}_c$ changes around x
  (why? Because the tangent to c is $\partial f / \partial x$ and its change is $\partial^2 f / \partial x^2$)

Curvature in 2D

- take a surface $S \subset \mathbf{R}^3$
- at each $x_0 \in S$
  - take a coordinate system xyz with x,y tangent to S and z along $\mathbf{n}_S$
  - locally, S can be described as a function $z = f(x,y)$

How to describe 2D curvature?

- 1D analogy: how quickly the normal $\mathbf{n}_S$ changes around x_0
- problem: we have a surface – in which direction to look for change?



We must compute

$$C(x,s) = \frac{\partial^2 f(x)}{\partial s^2}$$

for any direction s

The Curvature Tensor

$$C(x,s) = \frac{\partial^2 f(x)}{\partial s^2}$$

- recall our definition of a tensor $\mathbf{T} : \mathbf{R}^3 \times \mathbf{R}^3 \rightarrow \mathbf{R}$? The above is precisely that

Also note that

$$\frac{\partial^2 f}{\partial s^2}(x_0) = \mathbf{s}^T H \mathbf{s}$$

where H is the so-called Hessian of f

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

In other words, if we have H, we can compute the curvature tensor

- at any point x_0
- in any direction s

The Curvature Tensor

However, there's a problem with the previous definition

- we need to construct local coordinate systems at every point on S
- not obvious how to do that….

General solution:

Describe S as an implicit function (i.e. the zero-level isosurface of a function)

$$S = \{x \in \mathbf{R}^3 \mid f(x) = 0\}$$ for a given $f : \mathbf{R}^3 \to \mathbf{R}$

Then, we still have

$$\frac{\partial^2 f}{\partial s^2}(x_0) = \mathbf{s}^T H \mathbf{s}$$ where H is the 3x3 Hessian matrix $$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

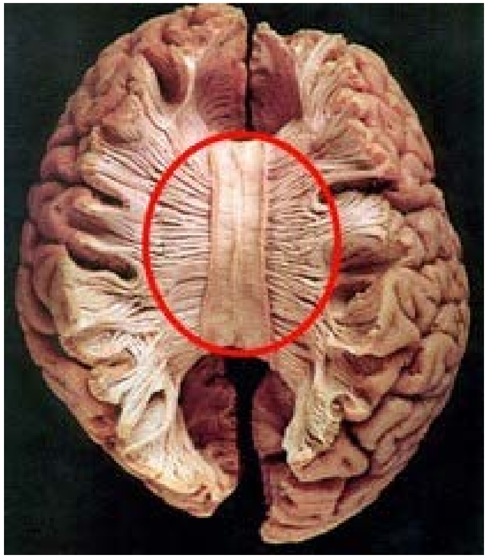
Conclusion

- A curvature tensor is fully described by a 3x3 matrix of 2nd order derivatives

The Diffusion Tensor

- consider an anisotropic material (e.g. tissue in the human brain)
- water diffuses in this tissue
  - strongly along neural fibers
  - weakly across fibers

Actual image of a dissected human brain



speed of water motion in tissue

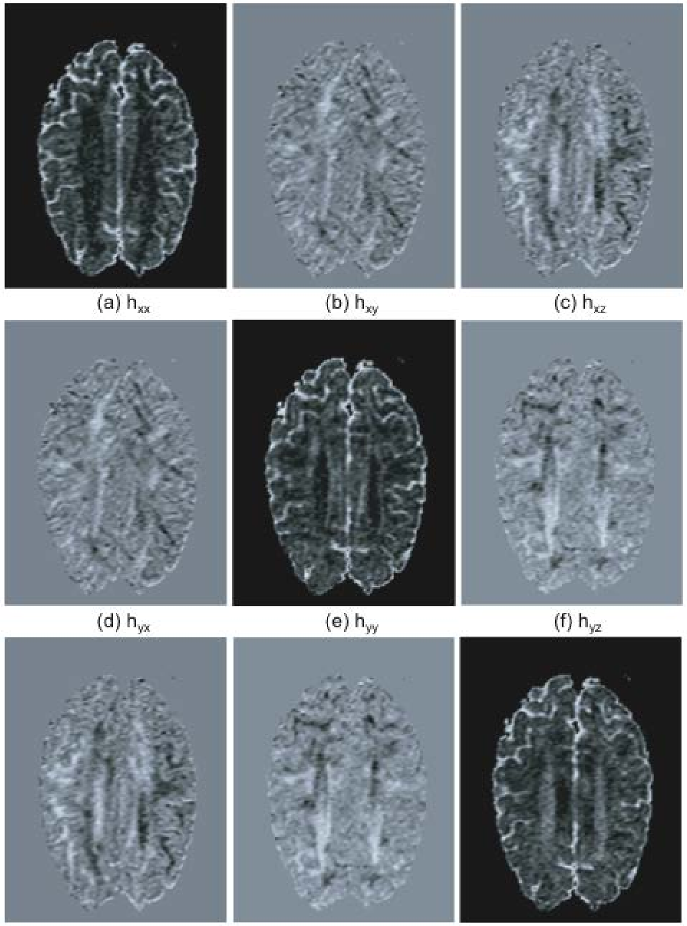
Diffusion tensor

$$D(x,s) = \frac{\partial^2 f(x)}{\partial s^2}$$

diffusivity at a point x in a direction s

Diffusion tensor: measured by a technique called **DT-MRI** (diffusion tensor magnetic resonance imaging)

The Diffusion Tensor



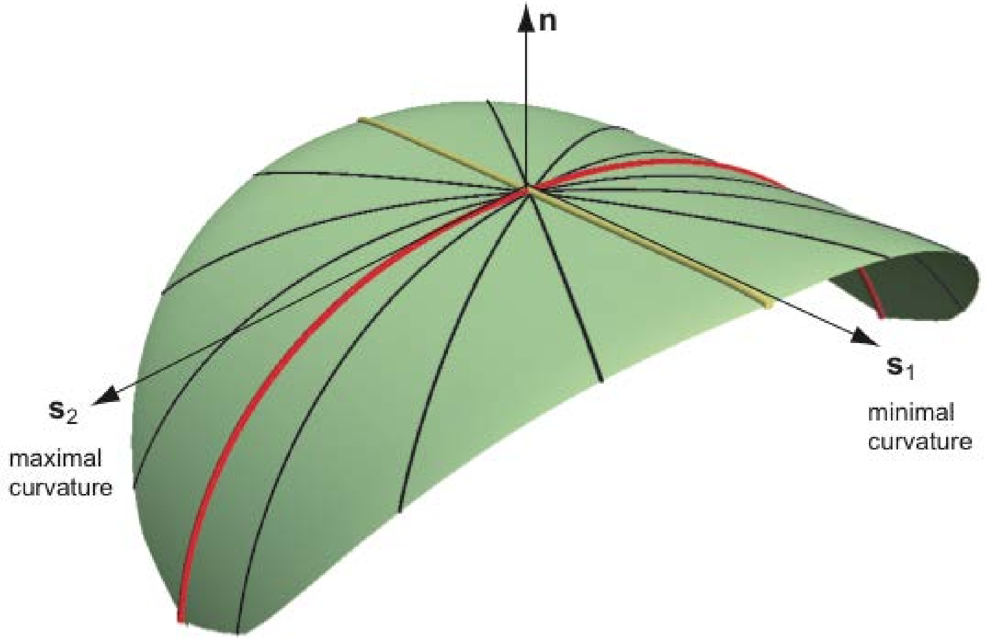
First visualization try

- compute hessian $H = \{h_{ij}\}$ in $\mathbf{R}^3$
- select some slice of interest
- visualize all components h_{ij} using e.g. color mapping

Simple, but not very useful

- we get a lot of images (9)…
- we see the tensor is symmetric…
- …but we don't really care about diffusion along x, y, z axes!

Principal Component Analysis



$$C(x,s)=\frac{\partial^2 f(x)}{\partial s^2}$$

- fix some point x_0 on the surface
- compute $C(x_0,s)$ for all possible tangent directions s at x_0
- denote α = angle of s with local coordinate axis x_0

So we have

$$\frac{\partial^2 f}{\partial s^2} = s^T H s = h_{11}\cos^2\alpha + (h_{12}+h_{21})\sin\alpha\cos\alpha + h_{22}\cos^2\alpha$$

Now, let's look for the values of α for which this function is extremal!

Principal Component Analysis

Our curvature (as function of α) is extremal when $\frac{\partial C}{\partial \alpha} = 0$

This is equivalent to a system of equations

$$\begin{cases} h_{11}\cos\alpha + h_{12}\sin\alpha & = \lambda\cos\alpha \\ h_{21}\cos\alpha + h_{22}\sin\alpha & = \lambda\sin\alpha, \end{cases}$$

which in matrix form is $H\mathbf{s} = \lambda\mathbf{s}$ or $(H - \lambda I)\mathbf{s} = \mathbf{0}$

Since we're looking for the non-trivial solution $\mathbf{s} \neq \mathbf{0}$ this means

$$\det(H - \lambda I) = (h_{11} - \lambda)(h_{22} - \lambda) - h_{12}h_{21} = 0$$

Solving the above 2nd order equation in λ yields

- two real values λ_1, λ_2 eigenvalues (principal values) of tensor

Plugging λ_1, λ_2 into $H\mathbf{s} = \lambda\mathbf{s}$ yields

- two direction vectors $\mathbf{s}_1$, $\mathbf{s}_2$ eigenvectors (principal directions) of tensor

Summarizing

- Given a 2x2 tensor, we can compute its principal directions and values
  - directions: those in which tensor has extremal (minimal, maximal) values
    Can be shown that eigendirections are orthogonal to each other
  - values: the actual minimal and maximal values

For full details, see Sec. 7.1

Principal Component Analysis

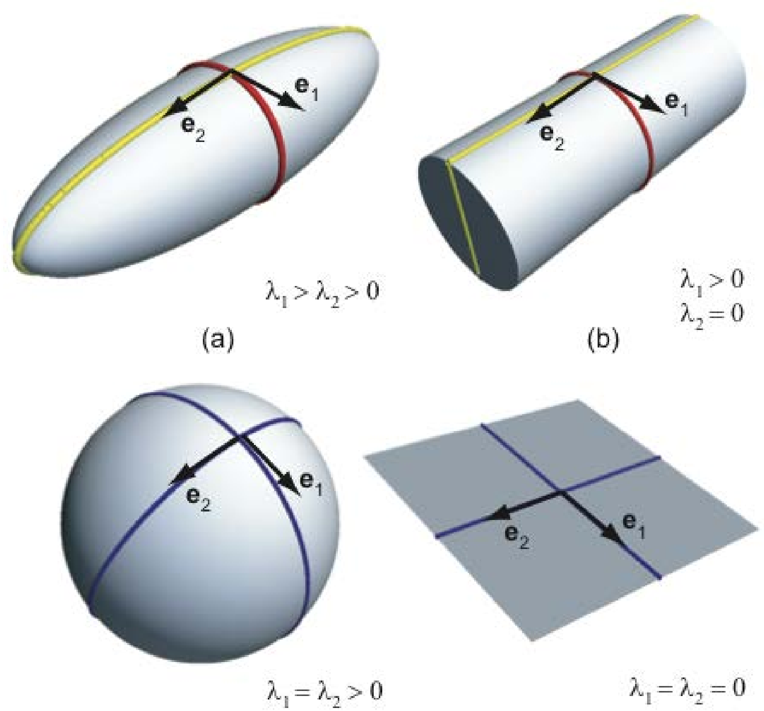
How about a 3x3 tensor, like the diffusion tensor?

- 3 eigenvalues, 3 eigenvectors (computed similarly, see Sec. 7.1)

Say we order eigenvalues (and their vectors) as $\lambda_1 > \lambda_2 > \lambda_3$

$\lambda_1, \mathbf{s}_1$ **major** eigenvector i.e. direction of strongest diffusion
$\lambda_2, \mathbf{s}_2$ **medium** eigenvector (no particular meaning)
$\lambda_3, \mathbf{s}_3$ **minor** eigenvector i.e. direction of weakest diffusion

What if two or more eigenvalues are equal (so we cannot fully order them all)?



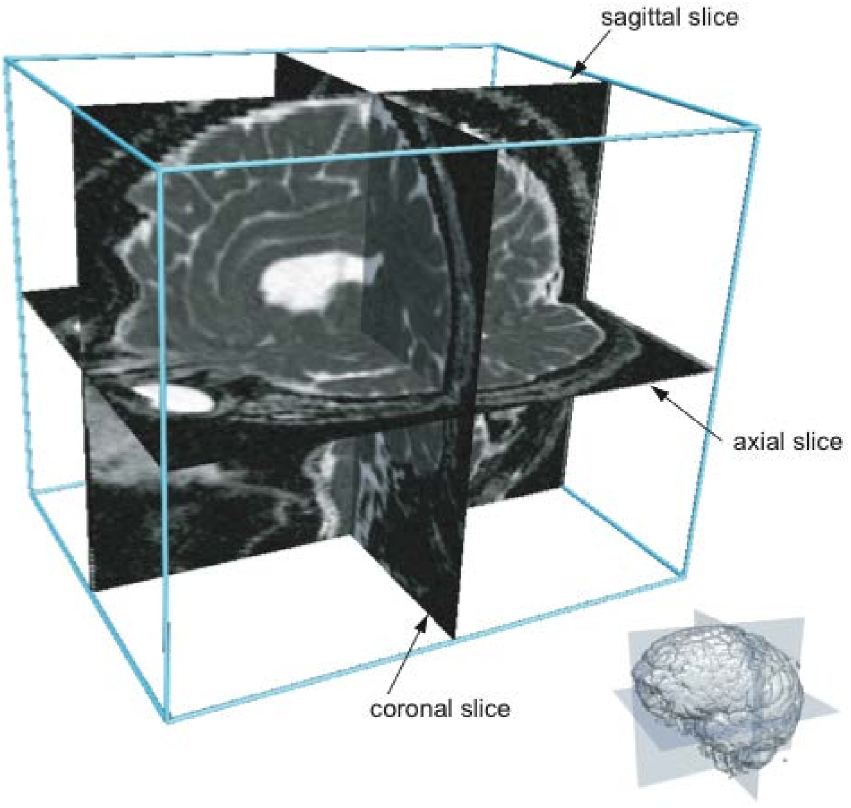
a,b) all values ordered: unique eigendirections

c,d) equal eigenvalues: eigendirections not determined (any two orthogonal vectors tangent to surface are valid eigendirections)

Principal Component Analysis

How to use PCA for visualization?

Visualize mean diffusivity $\mu = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)$



white: strong mean diffusivity
black: weak mean diffusivity

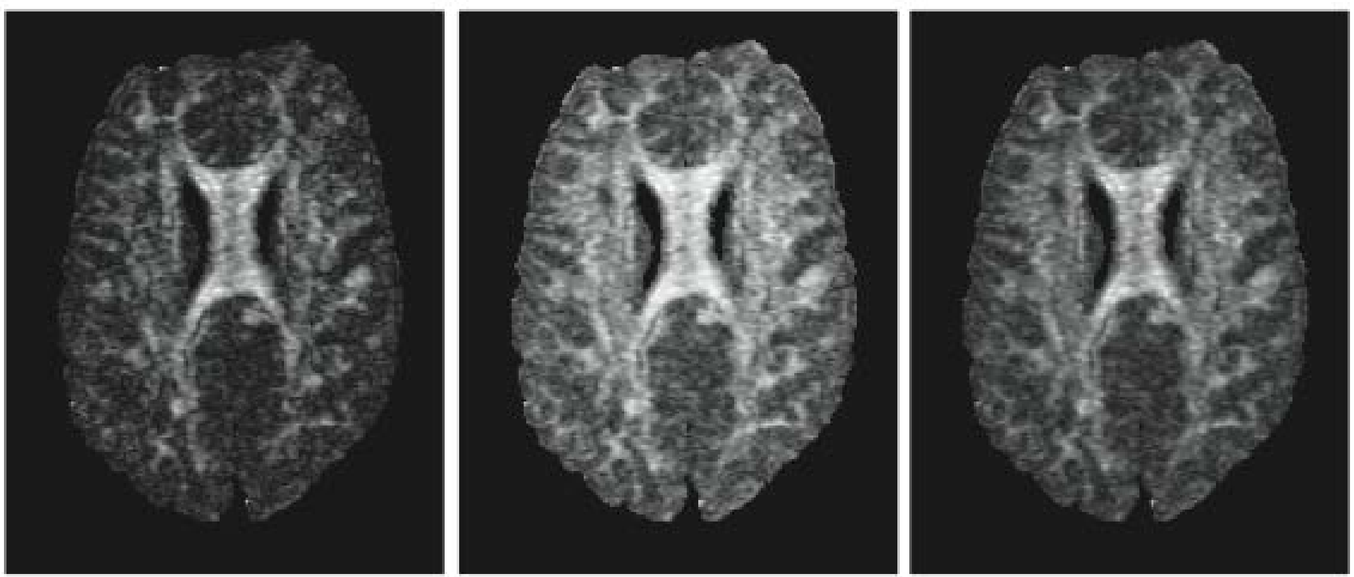
Principal Component Analysis

Linear diffusivity $$c_l = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}$$

Fractional anisotropy $$FA = \sqrt{\frac{3}{2}} \frac{\sqrt{\sum_{i=1}^{3} (\lambda_i - \mu)^2}}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$$ where $$\mu = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)$$

Relative anisotropy $$RA = \sqrt{\frac{3}{2}} \frac{\sqrt{\sum_{i=1}^{3} (\lambda_i - \mu)^2}}{\lambda_1 + \lambda_2 + \lambda_3}$$

All above measures estimate how much 'fiber-like' is the current point



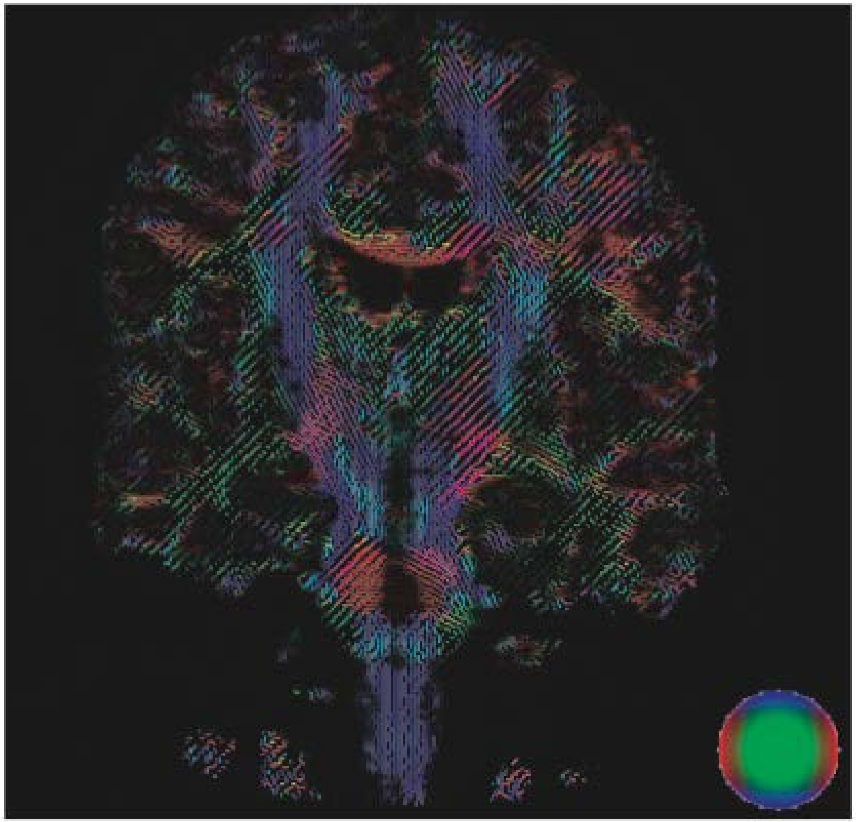
(a) c_l linear estimator (b) fractional anisotropy (c) relative anisotropy

white: strong fibers

Principal Component Analysis (PCA)

Exploit the directional information in the eigenvectors

- major eigenvector $\mathbf{e}_1$: along the **strongest** diffusion direction
- for DTI tensors, it thus indicates fiber directions



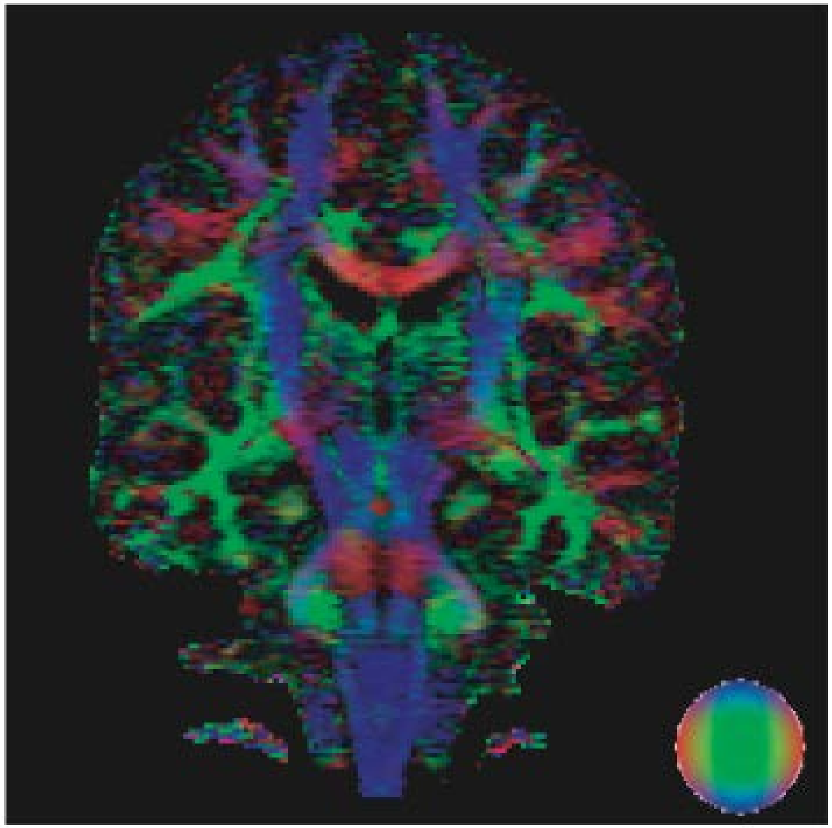
Directional color coding

- like for vectors (see Module 4)
- use simple colormap

$$R = |\mathbf{e}_1 \cdot \mathbf{x}|,$$
$$G = |\mathbf{e}_1 \cdot \mathbf{y}|,$$
$$B = |\mathbf{e}_1 \cdot \mathbf{z}|.$$

- use vector glyphs / hedgehogs
- seed only points where c_l, FA or RA are large enough (other points don't cover fibers)
- OK, but takes training to grasp

Vector PCA



Directional color coding (2nd variant)

- like before, but simply color points by direction
- no glyphs drawn
- no occlusion/clutter
- direction coded **only** by color – less intuitive images

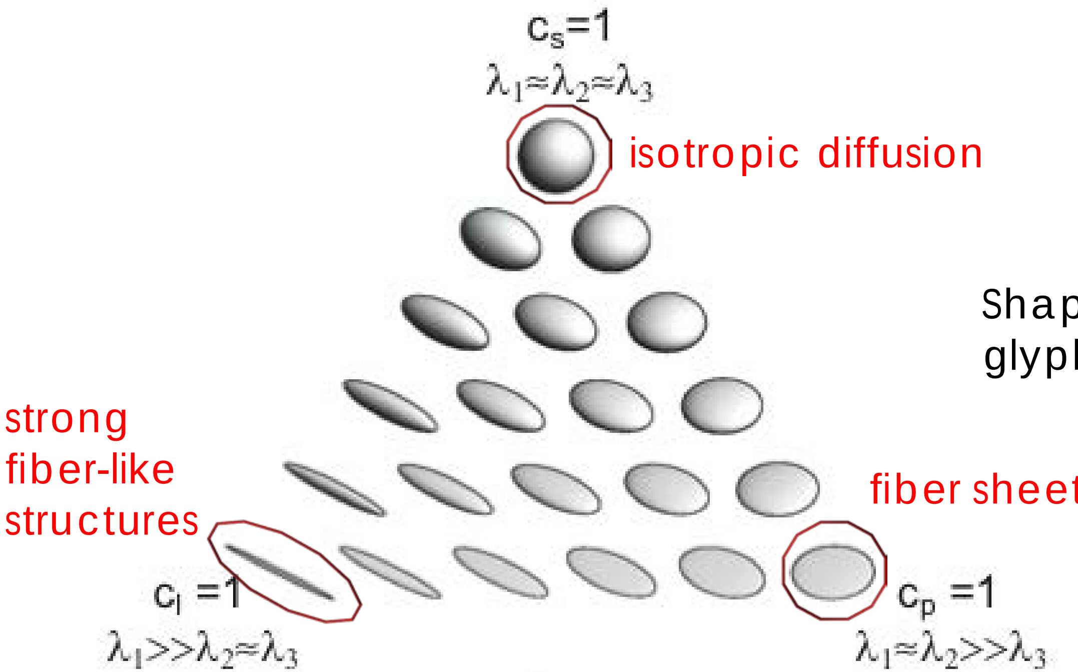
Tensor Glyphs

So far, we only visualized the major eigenvector $\mathbf{e}_1$

- so we reduced a tensor field to a vector field
- we **threw away** existing information (medium+minor eigenvectors $\mathbf{e}_2,\mathbf{e}_3$)

Ellipsoid glyph: Use all eigenvalues + eigenvectors

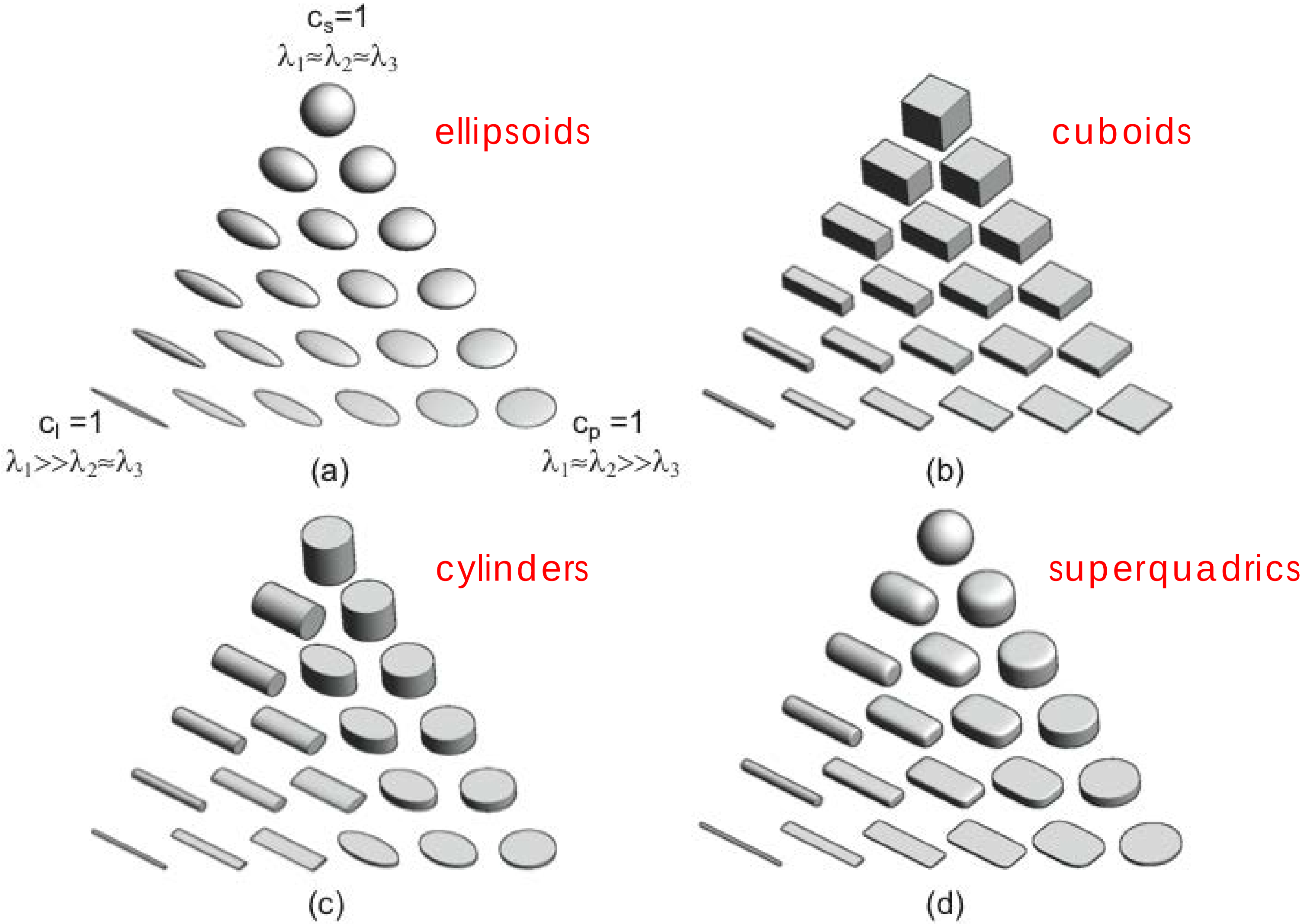
- orient glyph along eigensystem ($\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$)
- scale it by eigenvalues ($\lambda_1,\lambda_2,\lambda_3$)



Shapes that an ellipsoid glyph can assume

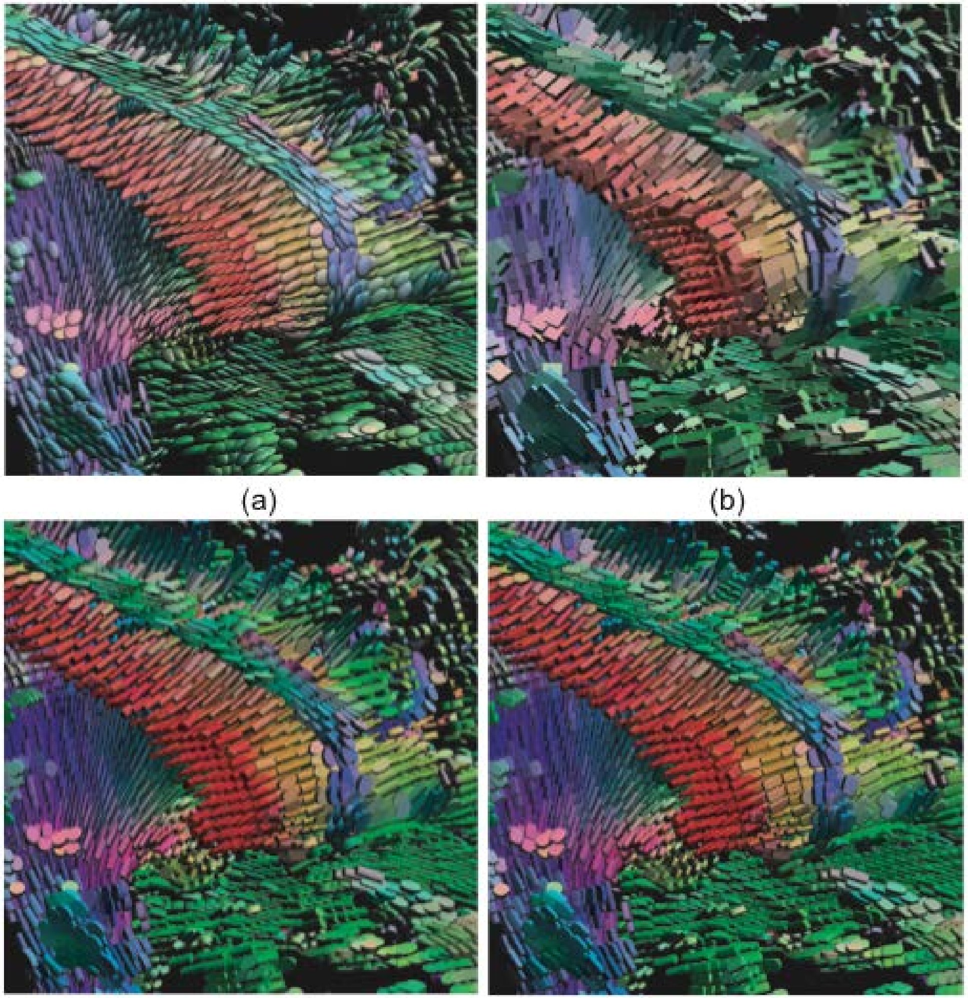
Tensor Glyphs

Can use other glyph shapes besides ellipsoids



Tensor Glyphs

Zoom-in on brain DT-MRI dataset



(a) (b) (c) (d)

a) ellipsoids
b) cuboids
c) cylinders
d) superquadrics

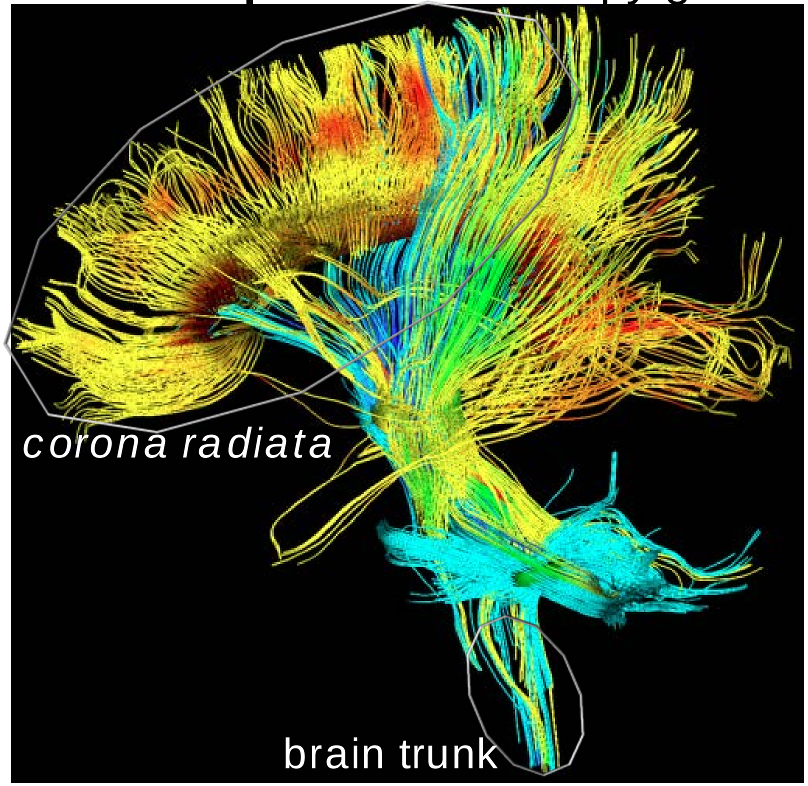
Superquadrics look arguably most 'natural'

For full details, see Sec. 7.5

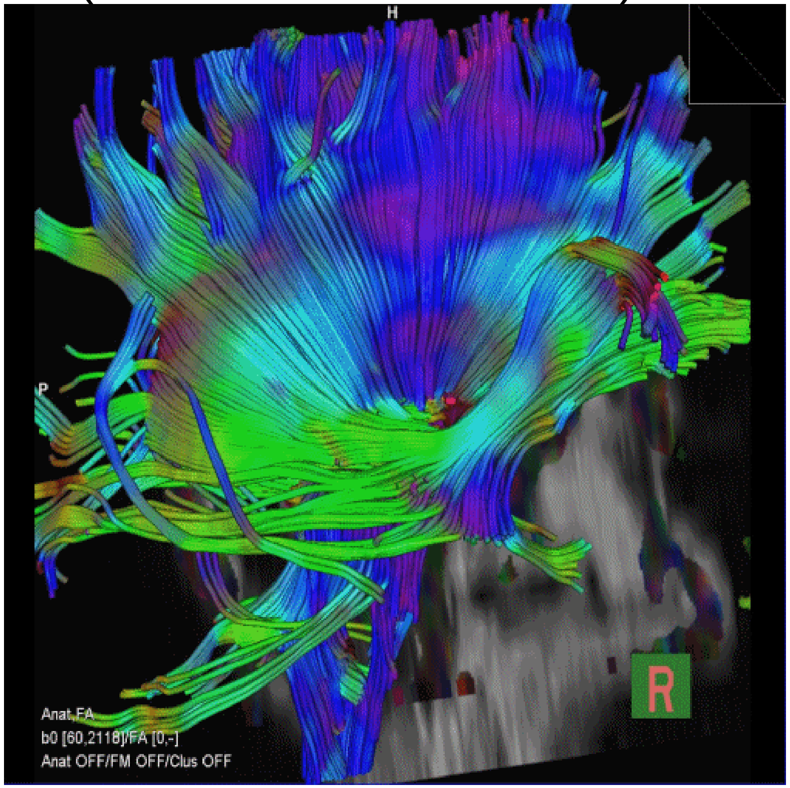
Fiber Tracking

Reuse some other vector visualization methods

- consider major eigenvector field
- trace streamlines
  - **seed:** in regions with high anisotropy (i.e. where fibers are)
  - **stop:** when anisotropy gets too low (i.e. when we leave fibers)



streamlines, brain overview



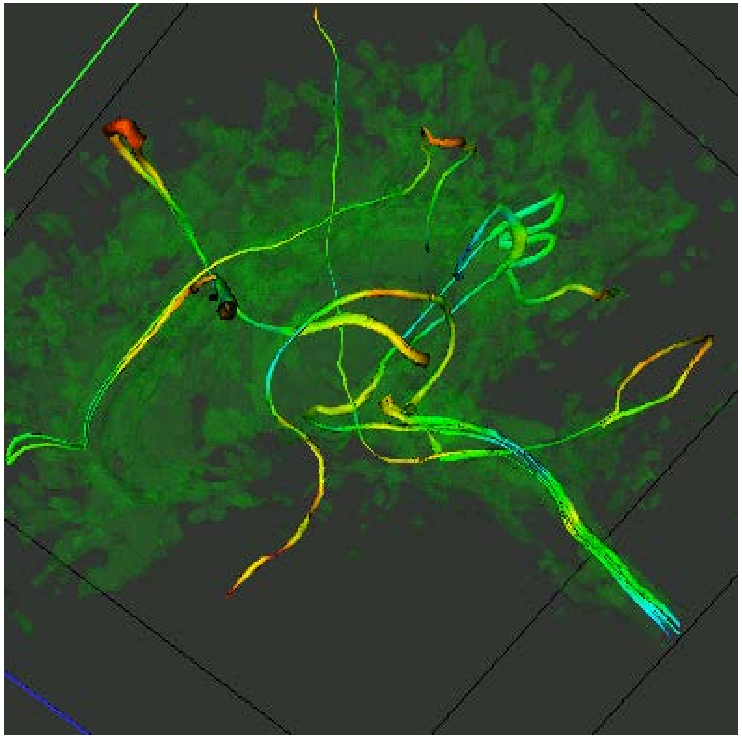
stream tubes, brain detail

This method is also called **tractography**

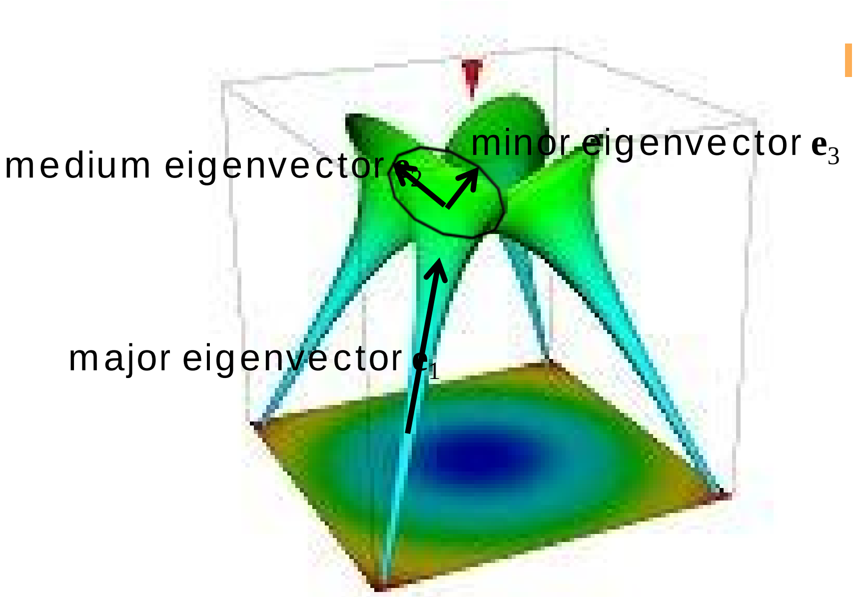
Hyperstreamlines

Generalize stream tubes

- trace stream tubes in major eigenvector field (like so far)
- use an **elliptic** cross-section
  - oriented along medium + minor eigenvectors
  - scaled with medium + minor eigenvalues



Idea



Tube cross-section shows diffusion across fibers

- Thin, round tubes: we're in a fiber **bundle**
- Thick, flat tubes: we're in a fiber **sheet**
- Thick, round tubes: we're **exiting** a fiber

Tensor Visualization Summary

- fundamentally harder than vector visualization
  - 9 values per point (!)
  - classical vector visualization problems (occlusion, seeding, etc)

- methods
  - reduce tensors to scalars (tensor components, PCA or anisotropies)
  - directional and/or color coding of major eigenvector
  - tensor glyphs
  - streamlines, stream tubes
  - hyperstreamlines