CS 498 VR

Lecture 7 - 2/7/18

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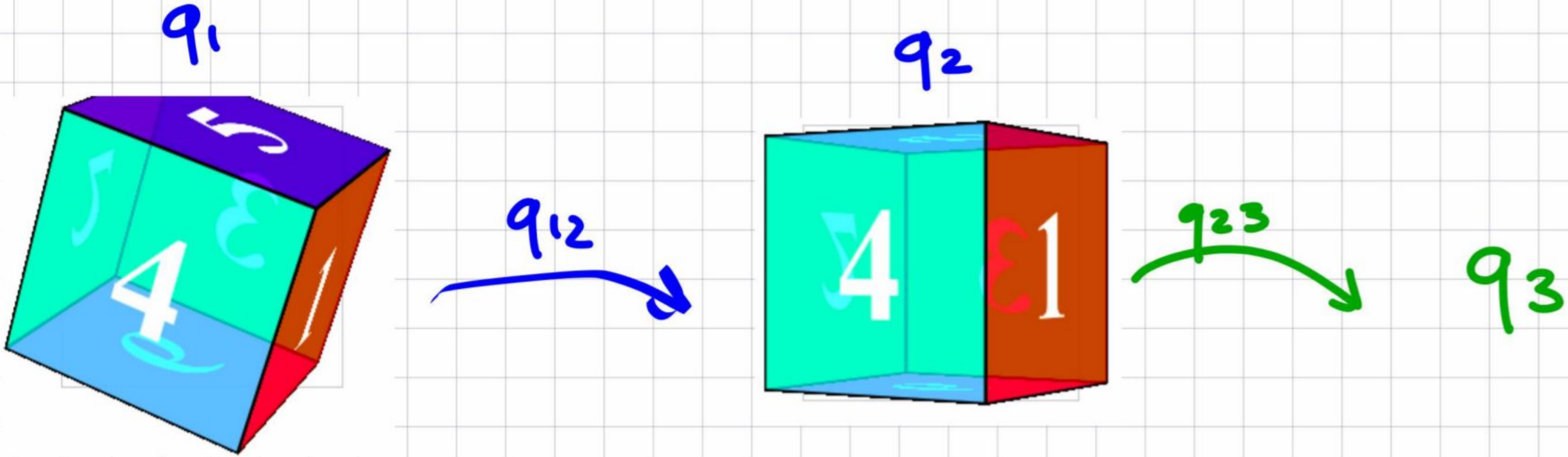
Last Time on CS 498

- What are the rotation matrices for pitch, yaw, and roll?
    - Exam question(s) detected.

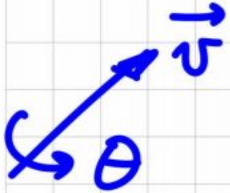
- In $M_F = M_c \cdot M_p$ which rotation matrix is applied first?

Unit Quaternions: Compositions, Inverses, Duplicates

$$q = \left(\cos\frac{\theta}{2},\ v_1 \cdot \sin\frac{\theta}{2},\ v_2 \cdot \sin\frac{\theta}{2},\ v_3 \cdot \sin\frac{\theta}{2}\right)$$



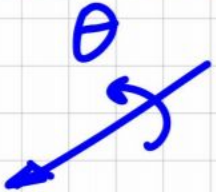
Unit Quaternions: Compositions, Inverses, Duplicates

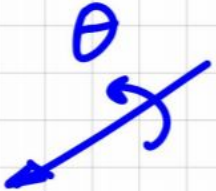


$$q = \left(\cos\frac{\theta}{2},\ v_1\cdot\sin\frac{\theta}{2},\ v_2\cdot\sin\frac{\theta}{2},\ v_3\cdot\sin\frac{\theta}{2}\right)$$

$$\Updownarrow$$

$$(\theta, \vec{v})$$





$(-\vec{v}, \qquad)$

$(v, \qquad)$

$q =$

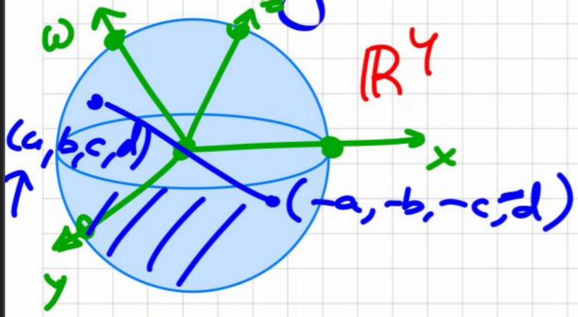
$q =$

Better!

Representation of Rotations: Unit Quaternions

$q = (a, b, c, d) \in \mathbb{R}^4$, $a^2 + b^2 + c^2 + d^2 = 1$

The set of all unit q is a hypersphere (S^3)



S^2 lives in $\mathbb{R}^3$

S^1 lives in $\mathbb{R}^2$

. . . .

S^n lives in $\mathbb{R}^{n+1}$

In Unity3D: $(x, y, z, w) = (b, c, d, a)$

In math: $a + bi + cj + dk$

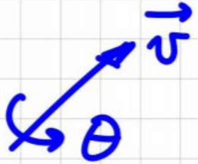
$\mathbb{R}^9$

Unit Quaternions: Compositions, Inverses, Duplicates

$$q = \left(\cos\frac{\theta}{2},\ v_1 \cdot \sin\frac{\theta}{2},\ v_2 \cdot \sin\frac{\theta}{2},\ v_3 \cdot \sin\frac{\theta}{2}\right)$$

$$\Updownarrow$$

$$(\theta, \vec{v})$$



$$q^{-1} =$$

Unit Quaternions: Compositions, Inverses, Duplicates

$$q = (\underbrace{\cos\frac{\theta}{2}}_{a},\ \underbrace{v_1 \cdot \sin\frac{\theta}{2}}_{b},\ \underbrace{v_2 \cdot \sin\frac{\theta}{2}}_{c},\ \underbrace{v_3 \cdot \sin\frac{\theta}{2}}_{d})$$

What is the inverse of q?

duplicate of q?

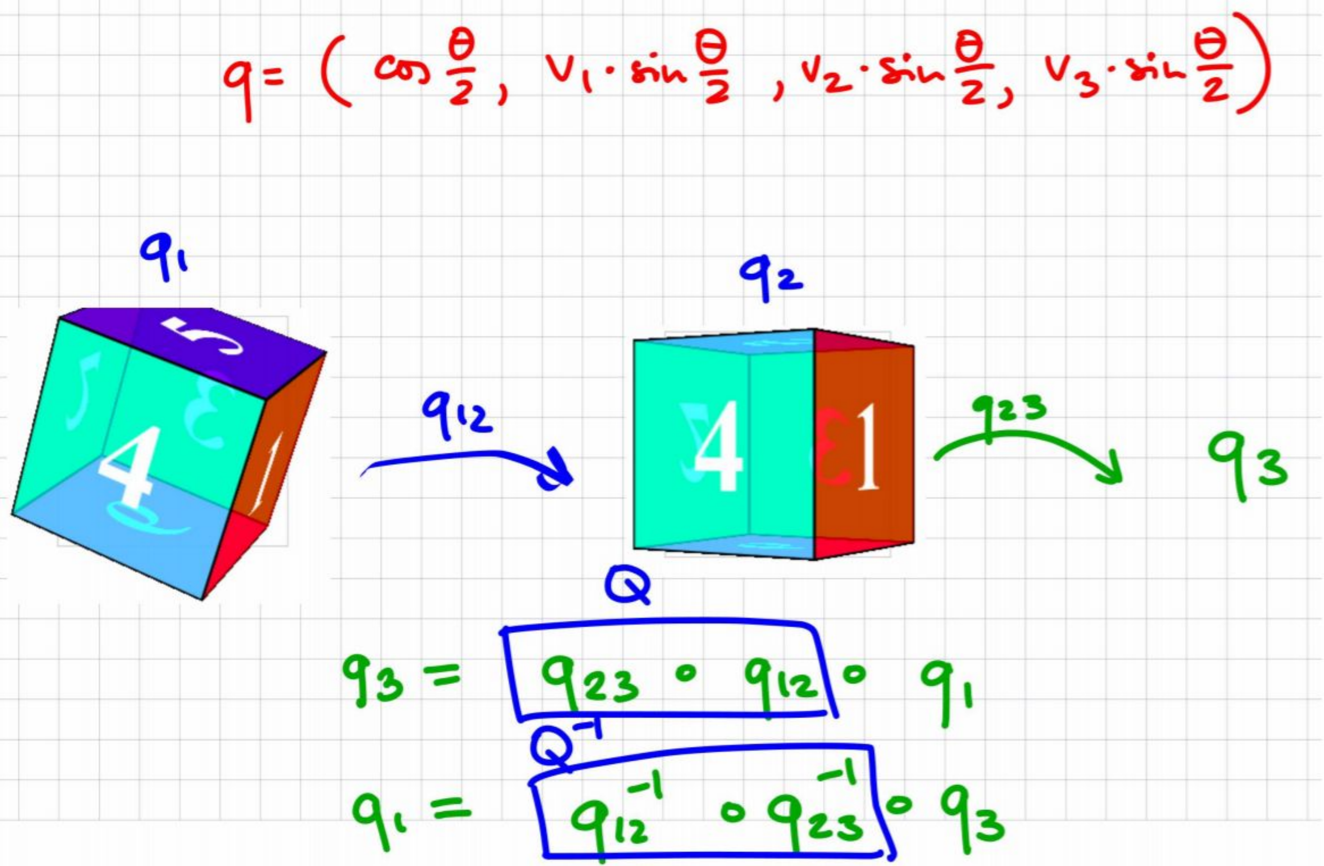
$$q = (a, b, c, d)$$

$$q \overset{?}{=} (-a, -b, -c, -d)$$

$$q \overset{?}{=} (-a, b, c, d)$$

$$q \overset{?}{=} (a, -b, -c, -d)$$

Unit Quaternions: Compositions, Inverses, Duplicates



Unit Quaternions: Multiplication

$q_1 = (a_1, b_1, c_1, d_1)$ $\vec{p}_1 = (b_1, c_1, d_1)$

$q_2 = (a_2, b_2, c_2, d_2)$ $\vec{p}_2 = (b_2, c_2, d_2)$

$$q_1 \circ q_2 = (a_1 a_2 - \vec{p}_1 \cdot \vec{p}_2 \,,\; \vec{p}_1 \times \vec{p}_2 + a_1 \vec{p}_2 + a_2 \vec{p}_1)$$

+renormalize

Order of operations! $q_1 \circ q_2 \neq q_2 \circ q_1$

Inverses $(q_4 \circ q_3 \circ q_2 \circ q_1)^{-1} =$

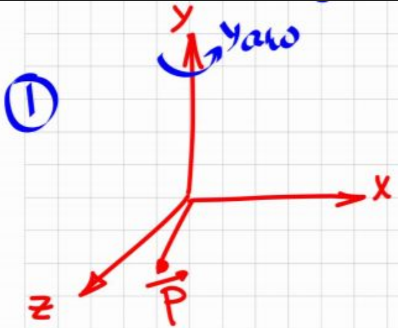
Efficiency: Haar Measure, only 4 parameters

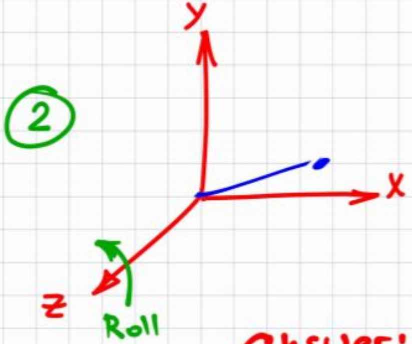
Character Matrices in Global Coordinate Frame

Steve is a Minecraft character. His head is a cube. The center of his head is the origin of the ***global*** coordinate frame, in which his left pupil has coordinates (1, 0, 3).

Calculate the coordinates of Steve's left pupil after Steve's head is turned first by a yaw of 90 degrees followed by a roll of 90 degrees in the ***global*** coordinate frame.

Chaining Matrices in Global Coordinate Frame





answer:

Pupil : $\vec{p} = (1, 0, 3)$, $p' = (\qquad\qquad)$, $p'' = (\qquad\qquad)$

① $R_y(\frac{\pi}{2}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ ② $R_z(\frac{\pi}{2}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} \qquad \\ \\ \\ \end{bmatrix} \begin{bmatrix} \qquad \\ \\ \\ \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} =$$

Applying Quaternion Rotation to a Vector

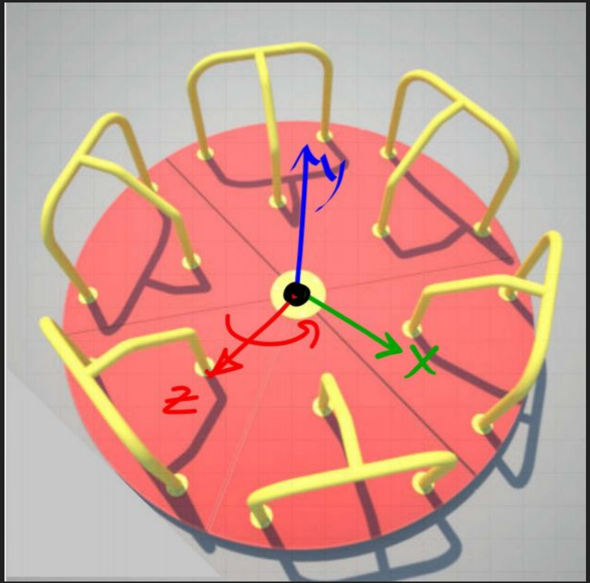
Vector $(x, y, z) \in \mathbb{R}^3$

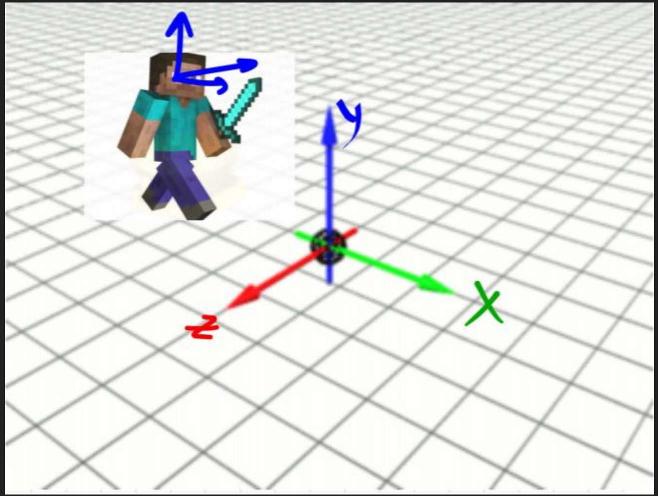
Rotate by quaternion q

$p = (x, y, z, 1)$

$p' = q \circ p \circ q^{-1}$

Characterizing Object Motion

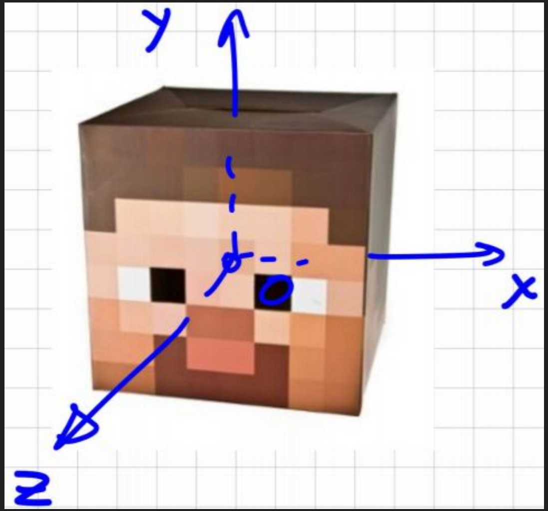




“Natural” with respect to local coord. frame.

“Natural” with respect to global coord. frame.

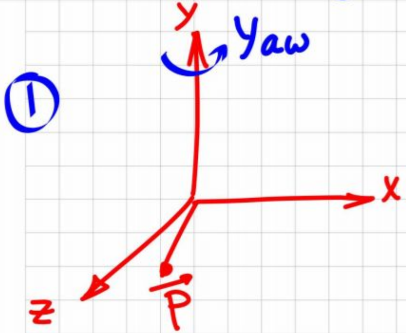
Another Formulation of the Same Problem



Steve is a Minecraft character. His head is a cube. Originally, his ***local*** coordinate frame coincides with the ***global*** coordinate frame and his left pupil has coordinates (1, 0, 3).

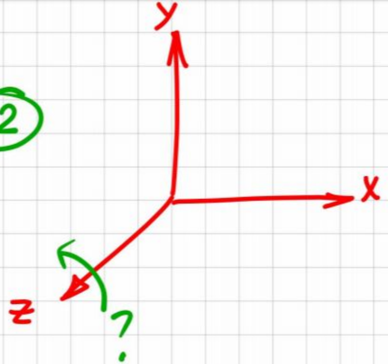
Calculate the coordinates of Steve’s left pupil after Steve turns his head first by a yaw of 90 degrees and then by a roll of 90 degrees in the ***global*** coordinate frame.

Chaining Matrices in Local Coordinate Frame



Intrinsic
vs
Extrinsic

②



$$① \; \underset{\text{yaw}}{R_y}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad ② \; R(\quad) = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

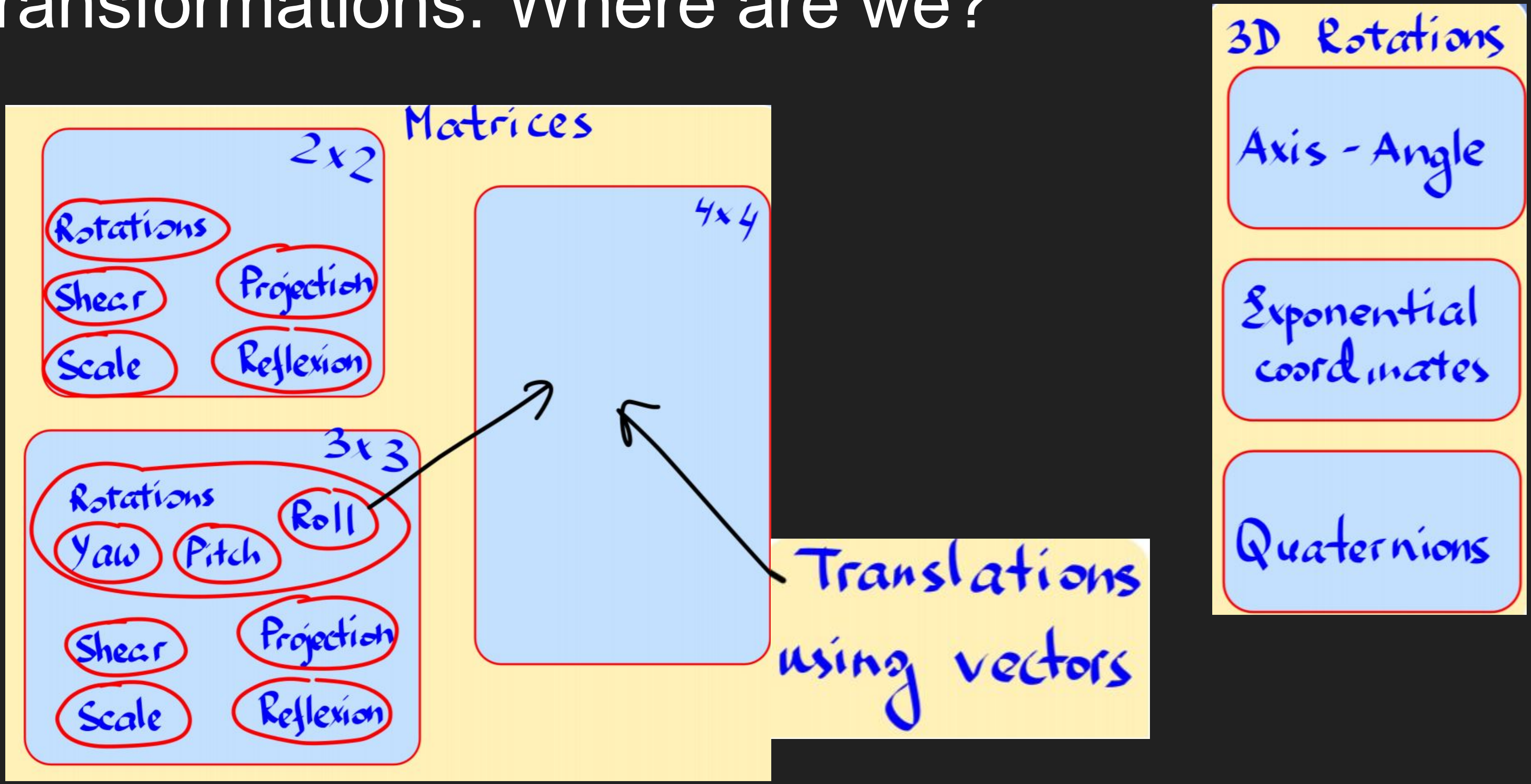
$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

Matrix Multiplication Property: Associativity

$$A \cdot B \cdot C \cdot D = A \cdot B \cdot C \cdot D$$

$$R_1 \cdot R_2 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_1 \cdot R_2 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Transformations: Where are we?



Limitations of 3x3 Matrices

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \ \end{bmatrix}$$

Chaining Translations and Rotations

Rotate by R, then
translate by $t = (t_x, t_y, t_z)$

$$\begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Place parenthesis in proper places

Translate by $t = (t_x, t_y, t_z)$,
then rotate by R

$$\begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Homogeneous Transformations: DOFs?

Rotate by R, then
translate by $\vec{t} = (t_x, t_y, t_z)$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = Rx + t \overset{?}{=} Ax$$

Describes ALL: possible rotations and translations of a 3D rigid body
(3D rigid transformations)

Sanity check:

$$\left.\begin{array}{l} \text{for } \vec{t} \\ \text{for } R \end{array}\right\} \text{total of}$$

Homogeneous Transformation Matrix

Rotate by R, then translate by $t = (t_x, t_y, t_z)$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Have: $x' = Rx + t$ Want: $x' = Ax$

Solution: Algebraic trick

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \\ \end{bmatrix}$$

Review

- Give an example of two different unit quaternions that result in the same rotation.
- Invert the quaternion $Q = (a, b, c, d)$.
    - Now give a duplicate of the inversion.
- If $Q = (a, b, c, d)$ is a **unit quaternion**, what constraint can we put on $a, b, c,$ and d (what do we know about them)?
- Explain in English (or your prefered verbal language) why translations and rotations are **not** commutative.

Announcements

- MP 2.1 due Monday @ 4:00 PM.
    - … The rest’s due the Monday after that.
- Team Formation Survey (Piazza) **also** due Monday
    - Do it or your MP grade will **suffer**.

- Read Ch. 3.2 & 3.3