CS 498 VR

Lecture 6 - 2/05/2018

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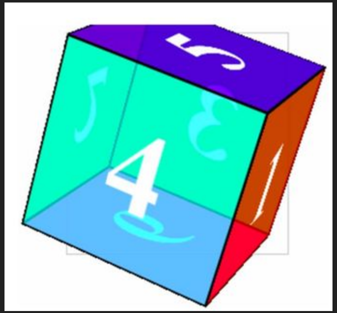
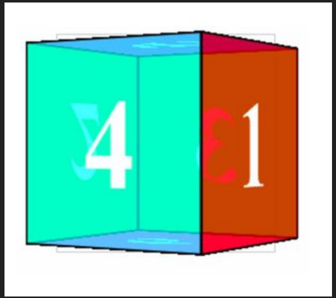
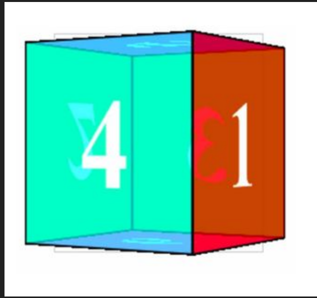
2D and 3D Matrices: Linear Transformations





[3D Picture Box]

2D Linear Transformations: Compositions



2D Linear Transformations: Compositions

1. Order of multiplication matters?

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

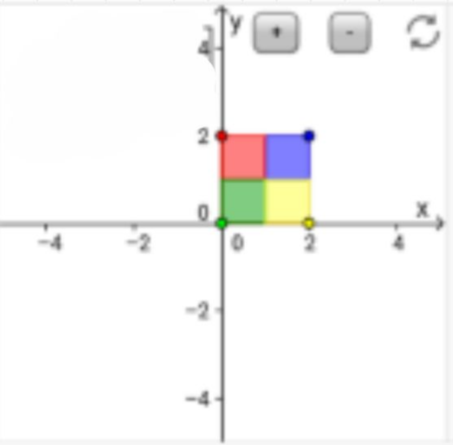
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

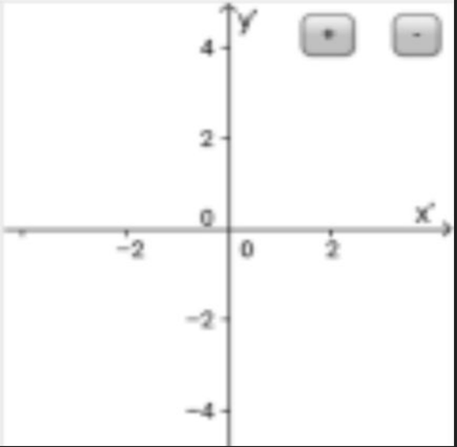
2. Which one gets applied first

$$M_3 = M_2 \cdot M_1$$

[Linear Transformation online toy]

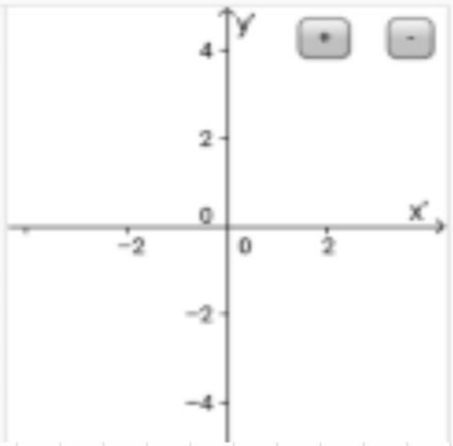
2D Linear Transformations: Review





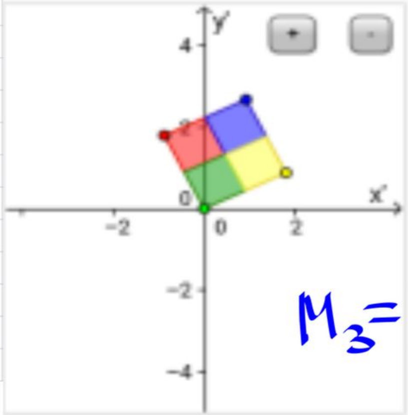
Draw:

$$M_1 = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}$$



Draw:

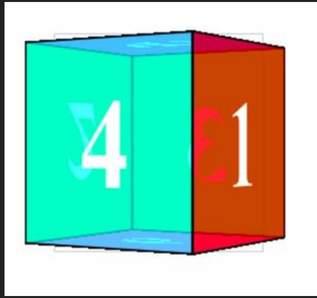
$$M_2 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$



Find rotation:

$$M_3 = \begin{bmatrix} \ \end{bmatrix}$$

2D Linear Transformations: Inverse



2D Linear Transformations: Inverse

Def:

Stretch: $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} & \\ & \end{bmatrix}$

Shear: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} & \\ & \end{bmatrix}$

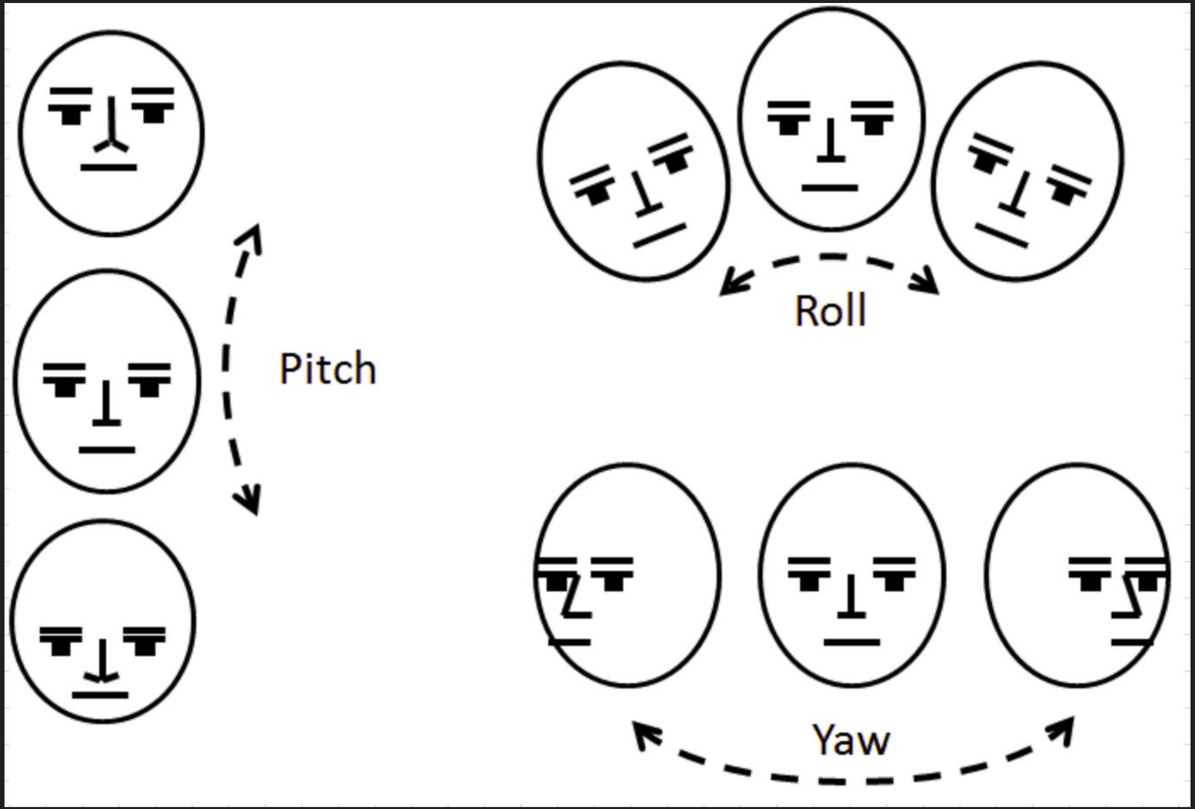
Rotation: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1}$

Composition: $(M_4 \cdot M_3 \cdot M_2 \cdot M_1)^{-1} =$

Rigid Body Transformations
Degrees of Freedom?

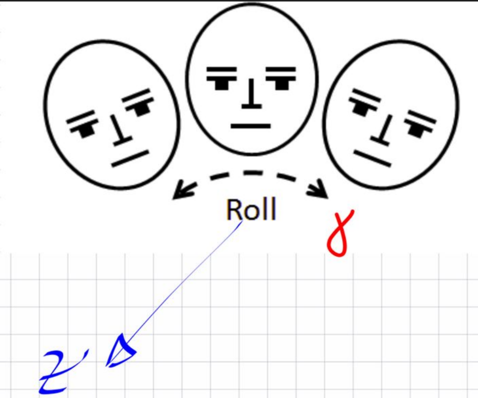
	2D	3D
Easy Translation	2	3
More Difficult Rotation	1	3
Most Difficult Rotation + Translation	3	6

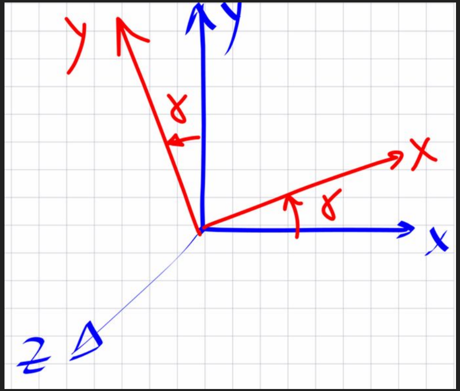
3D Canonical Rotations: Yaw, Pitch, Roll

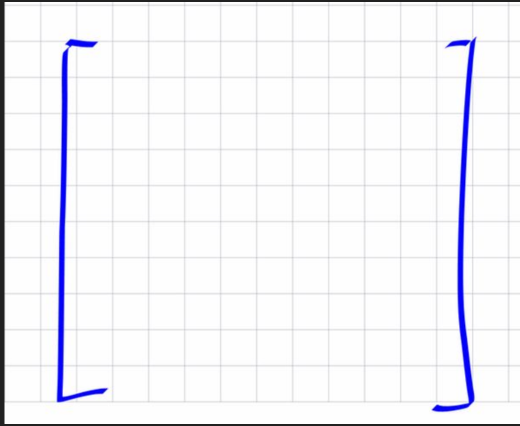


Also called Euler angles

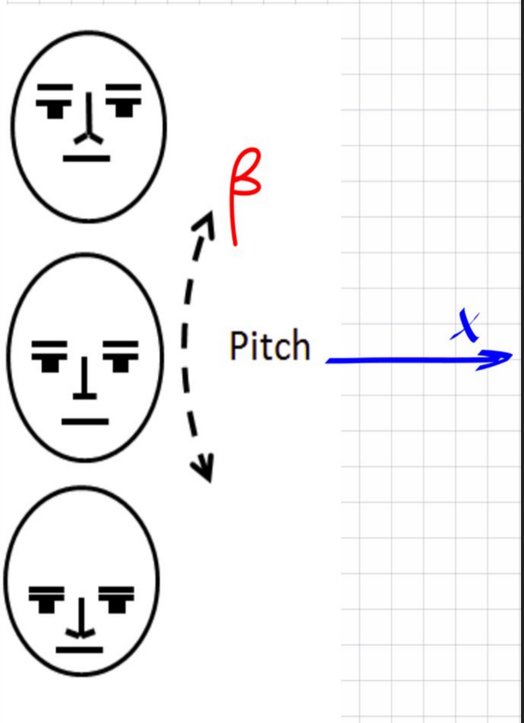
3D Canonical Rotations: Roll

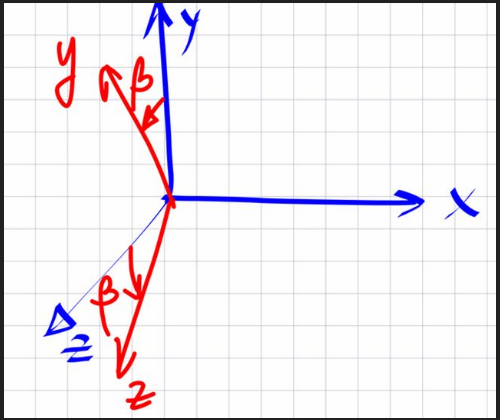




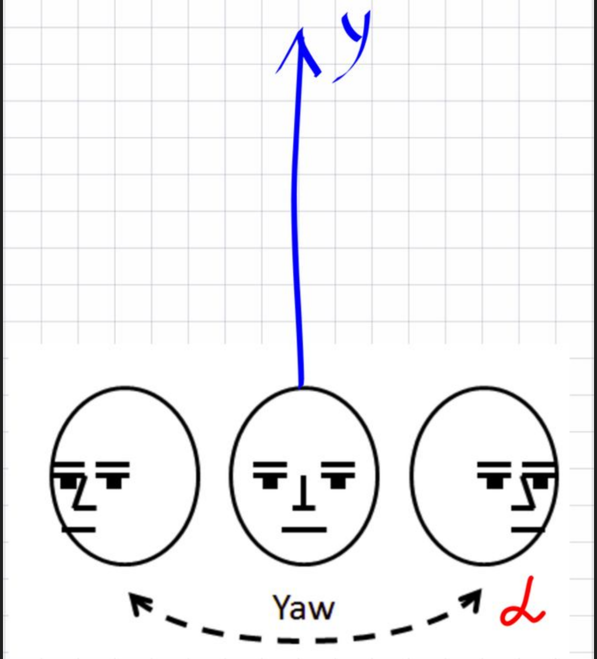


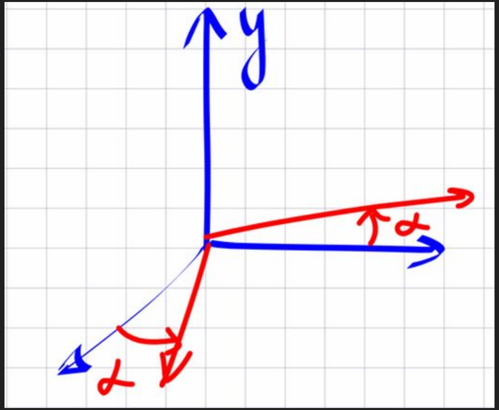
3D Canonical Rotations: Pitch

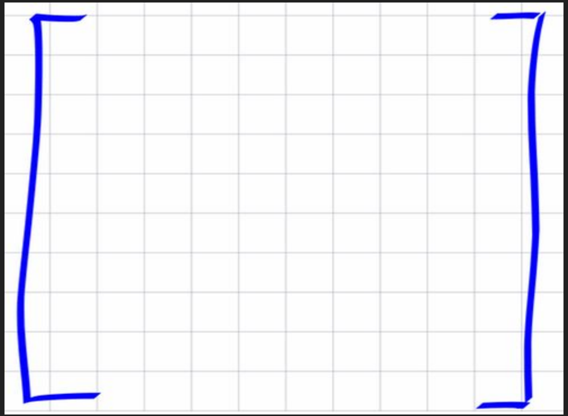




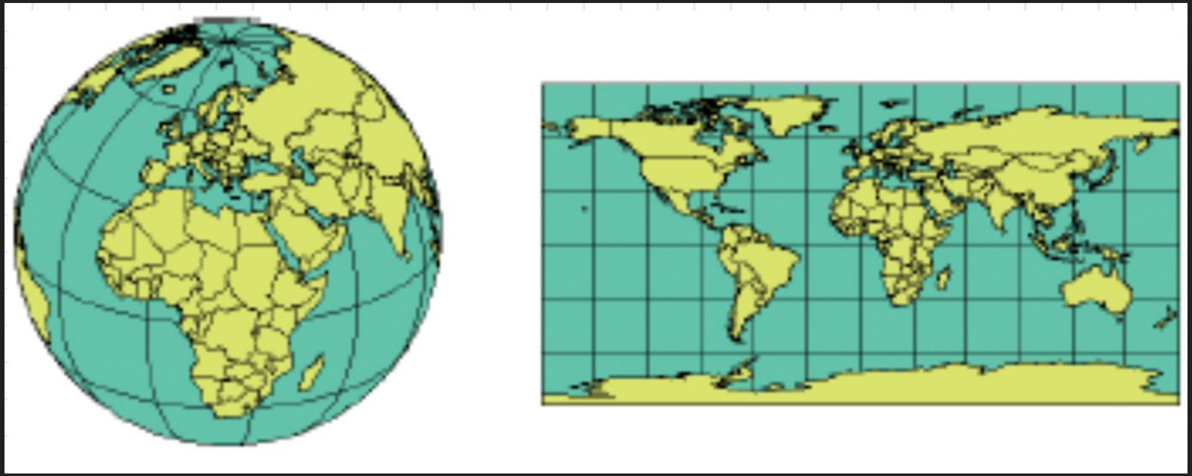
3D Canonical Rotations: Yaw







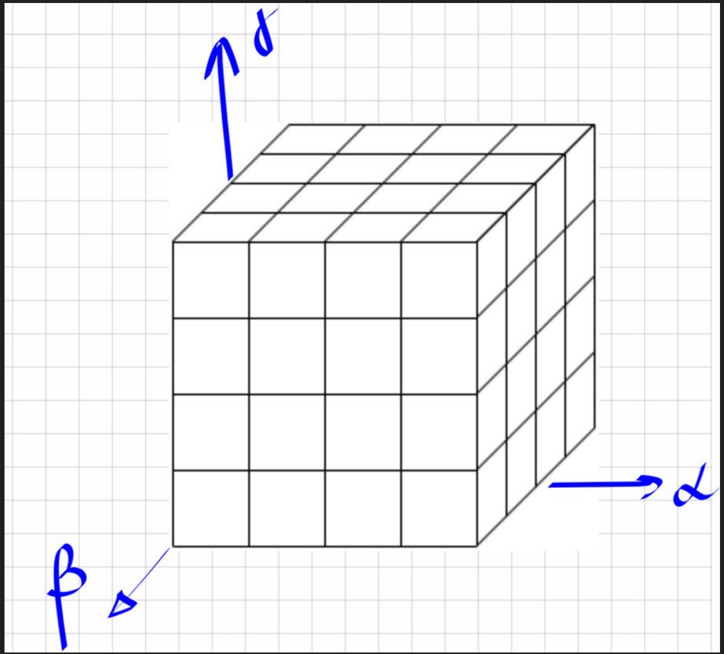
Parameterizing a 2D Sphere



1. Singularities

2. Shortest distances

Parameterizing Rotations: Yaw, Pitch, Roll



1. Singularities

2. Shortest distances

Parameterizing Rotations: Yaw, Pitch, Roll

1. Commutativity

$$\underbrace{R_y\left(\frac{\pi}{2}\right)}_{\text{yaw}} \cdot \underbrace{R_z\left(\frac{\pi}{2}\right)}_{\text{roll}}$$

$$\underbrace{R_z\left(\frac{\pi}{2}\right)}_{\text{roll}} \cdot \underbrace{R_y\left(\frac{\pi}{2}\right)}_{\text{yaw}}$$

2. Singularity (Gimbal Lock):

$$R_y(\alpha) \cdot R_x\left(\frac{\pi}{2}\right) \cdot R_z(\gamma) =$$

$$\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Representing Rotations: Euler's Rotation Theorem

Theorem:

1. Axis-Angle Representation:

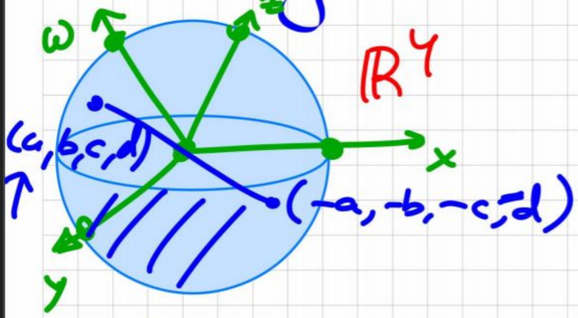
2. Unit Quaternions:

Better!

Representation of Rotations: Unit Quaternions

$q = (a, b, c, d) \in \mathbb{R}^4$, $a^2+b^2+c^2+d^2=1$

The set of all unit q is a hypersphere (S^3)



S^2 lives in $\mathbb{R}^3$

S^1 lives in $\mathbb{R}^2$

....

S^n lives in $\mathbb{R}^{n+1}$

In Unity3D: $(x, y, z, w) = (b, c, d, a)$

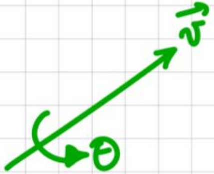
In math: $a+bi+cj+dk$

$\mathbb{R}^9$

From Axis-Angle to Unit Quaternions

Axis-angle: $(\theta, \vec{v})$

$\vec{v} = (v_1, v_2, v_3)$ $\quad \|\vec{v}\| = 1$



$v_1^2 + v_2^2 + v_3^2 = 1$

Corresponding unit quaternion:

$$q = (\underbrace{\cos\frac{\theta}{2}}_{a}, \underbrace{-\sin\frac{\theta}{2}\, v_1}_{b}, \underbrace{-\sin\frac{\theta}{2}\, v_2}_{c}, \underbrace{-\sin\frac{\theta}{2}\, v_3}_{d})$$

Sanity check: is this q unit?

$$\underbrace{\cos^2\frac{\theta}{2}}_{a^2} + \underbrace{\sin^2\frac{\theta}{2}\, v_1^2}_{b^2} + \underbrace{\sin^2\frac{\theta}{2}\, v_2^2}_{c^2} + \underbrace{\sin^2\frac{\theta}{3}\, v_3^2}_{d^2} \stackrel{?}{=} 1$$

Unit Quaternions Examples

$$q = \left(\cos\frac{\theta}{2},\ v_1 \cdot \sin\frac{\theta}{2},\ v_2 \cdot \sin\frac{\theta}{2},\ v_3 \cdot \sin\frac{\theta}{2} \right)$$

$(0, 1, 0, 0)$

$\left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)$

$(0, 0, 1, 0)$

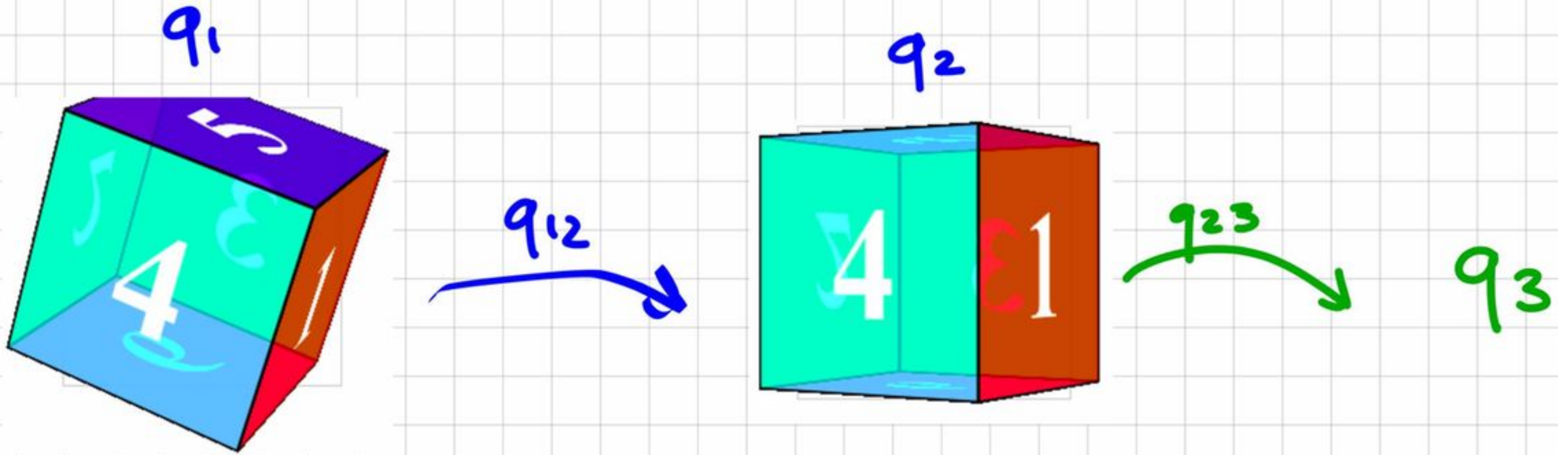
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$

$(0, 1, 0, 0)$

$(1, 0, 0, 0)$

Unit Quaternions: Compositions, Inverses and Duplicates

$$q = \left(\cos\frac{\theta}{2},\ v_1 \cdot \sin\frac{\theta}{2},\ v_2 \cdot \sin\frac{\theta}{2},\ v_3 \cdot \sin\frac{\theta}{2} \right)$$



Unit Quaternions: Compositions, Inverses and Duplicates

$$q = \left(\cos\frac{\theta}{2},\ v_1 \cdot \sin\frac{\theta}{2},\ v_2 \cdot \sin\frac{\theta}{2},\ v_3 \cdot \sin\frac{\theta}{2} \right)$$

What is the inverse of q?

Unit Quaternions: Multiplication

$q_1 = (a_1, b_1, c_1, d_1)$ $\vec{p}_1 = (b_1, c_1, d_1)$

$q_2 = (a_2, b_2, c_2, d_2)$ $\vec{p}_2 = (b_2, c_2, d_2)$

$$q_1 \circ q_2 = (a_1 a_2 - \vec{p}_1 \cdot \vec{p}_2 \,,\; \vec{p}_1 \times \vec{p}_2 + a_1 \vec{p}_2 + a_2 \vec{p}_1)$$

Order of operations! $q_1 \circ q_2 \neq q_2 \circ q_1$

Inverses $(q_4 \circ q_3 \circ q_2 \circ q_1)^{-1} = q_1^{-1} \circ q_2^{-1} \circ q_3^{-1} \circ q_4^{-1}$

Efficiency: Haar Measure, only 4 parameters

Homework

- Lavalle, CH 3.2, 3.3
- Survey for class projects