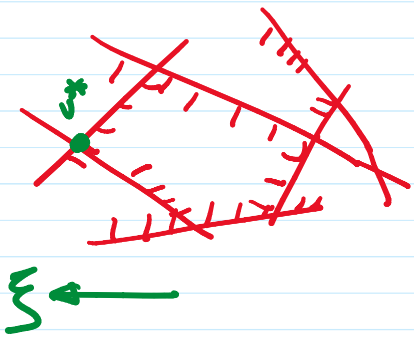
October 17, 2024 3:13 PM

LP in 2D

Given n halfspaces in $\mathbb{R}^2$
& vector ξ,
find a point in intersection
extreme along ξ.



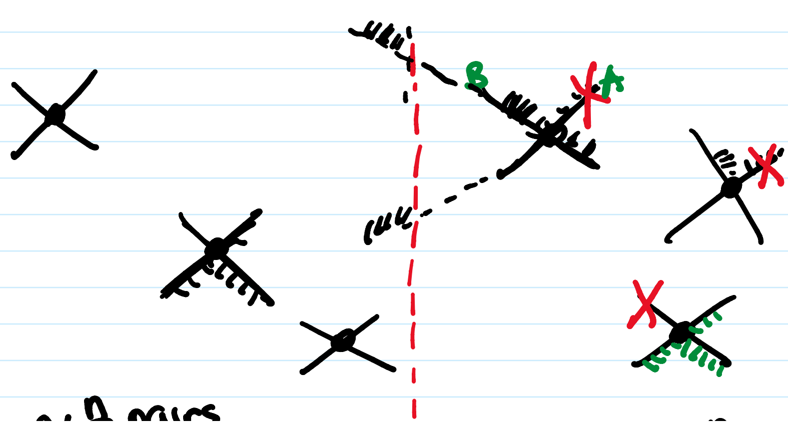
Prune-and-Search (Megiddo '83 / Dyer '83)

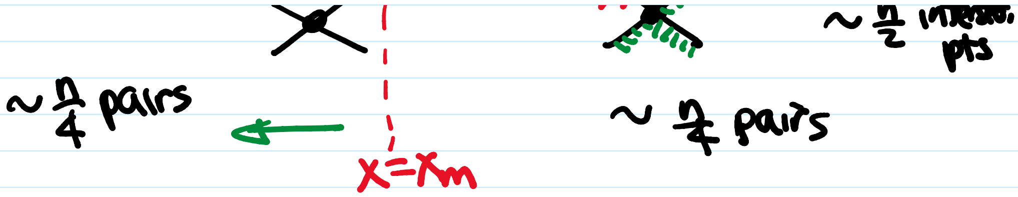
let L^+ = lines for upper halfplanes
L^- = lines for lower halfplanes
v^* = optimal sol'n

repeat {

1. pair up lines in L^+, lines in L^-
2. intersect each pair
3. compute median x-coord x_m of these $n/2$ intersection pts

O(n) time →





"decision test" $O(n)$ time? ↗ 4. if v^* left of $x = x_m$ then

for each pair with intersection pt in $x > x_m$

remove line w. larger slope if from L^+

- - - smaller " if from L^-

else similar

}

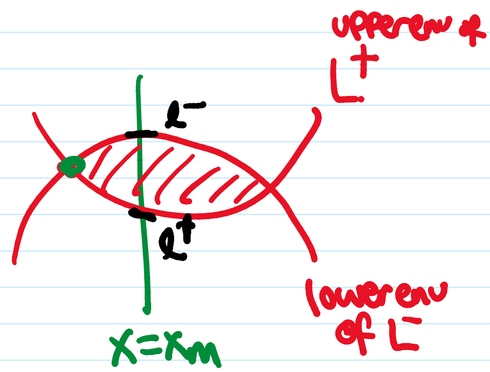
$$\Rightarrow \quad T(n) \le T\left(\tfrac{3}{4}n\right) + O(n)$$

$$\Rightarrow \quad O\left(n + \tfrac{3}{4}n + \left(\tfrac{3}{4}\right)^2 n + \dots\right)$$

$$= O\left(\frac{1}{1-\frac{3}{4}} n\right) = O(n).$$

How to decide in line 4?

say objective is to min x



4.1. find highest line $\ell^+ \in L^+$

lowest line $\ell^- \in L^-$

at $x = x_m$

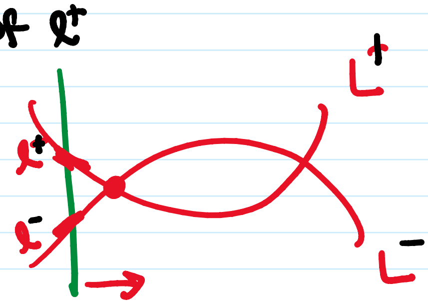
$O(n)$ ↗

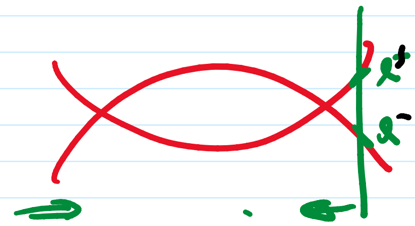
4.2. if ℓ^- above ℓ^+ at $x = x_m$

or slope of ℓ^- < slope of ℓ^+

then left

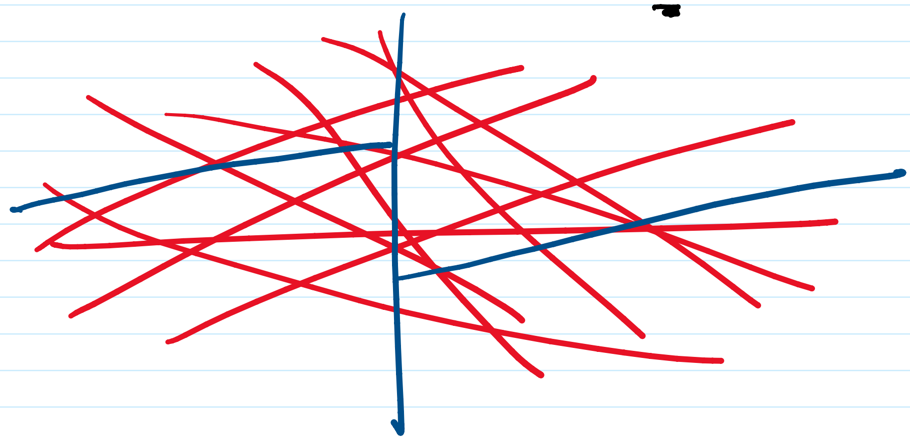
else right





Extension to 3D (Megiddo '84)

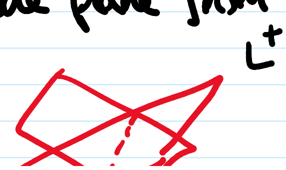
Cutting Lemma Given n lines in $\mathbb{R}^2$, can divide $\mathbb{R}^2$ into 4 regions s.t. each region intersects $\leq \frac{7}{8}n$ lines

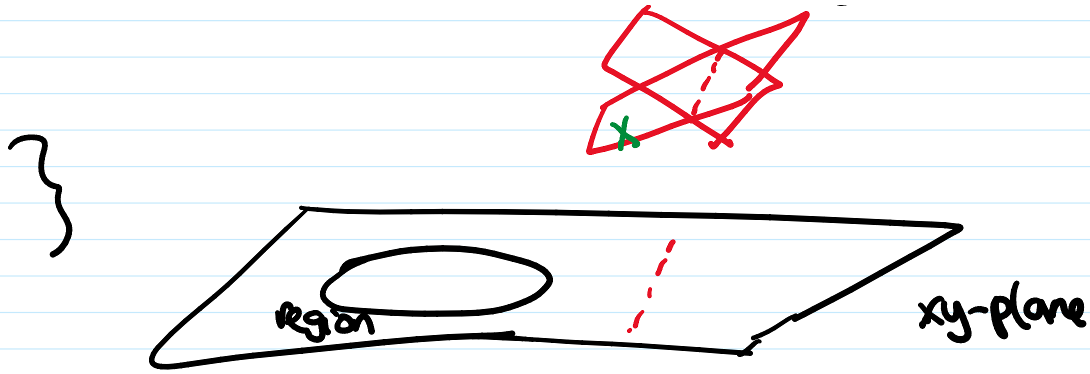


The algm:

repeat {

1. pair up planes in L^+ & planes in L^-
2. intersect each pair
3. apply Cutting Lemma to xy-proj of these $n/2$ lines
4. decide which region v^* is in
5. for each pair whose intersection line avoids the region
   remove one plane from pair





$$\Rightarrow T(n) \leq T\left(\frac{15n}{16}\right) + O(n)$$

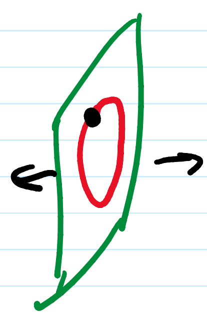
$$1 + \frac{15}{16} + \left(\frac{15}{16}\right)^2 + \ldots$$

$$\Rightarrow \boxed{O(n)}$$

How to decide:

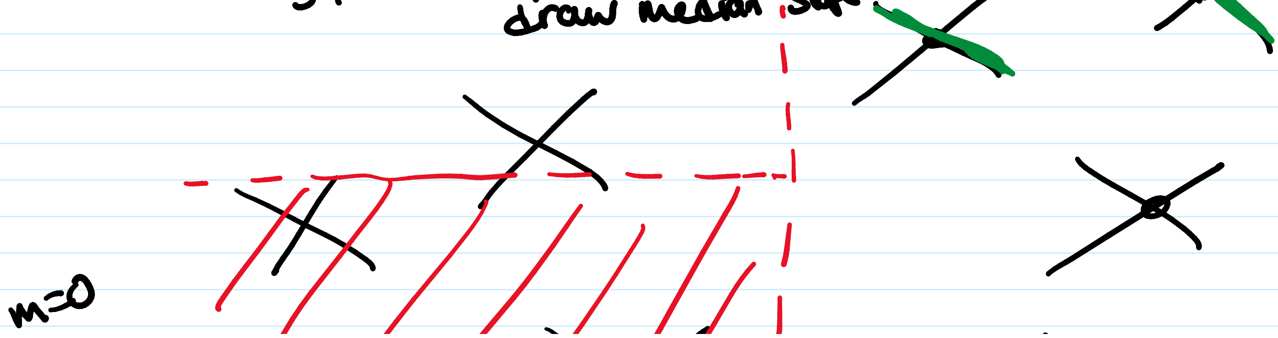
Solve 2D LP ...

$O(n)$ time

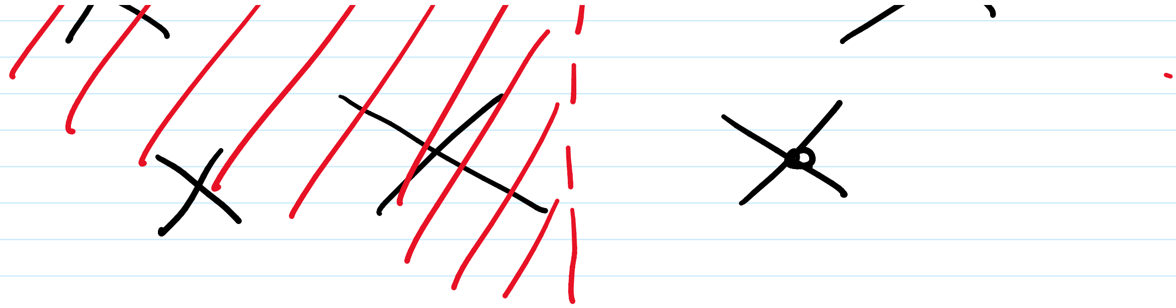


Pf of Cutting Lemma:

1. median slope m
2. pair lines of slope $< m$ w. lines of slope $> m$
3. Intersect each pair
4. median vertical line
5. on left side: draw median slope-m line thru pts to the right



$m=0$



Megiddo $O(2^{O(2^d)} n)$

Dyer $O(2^{O(d^2)} n)$

current best deterministic $O(2^{O(d \log d)} n)$