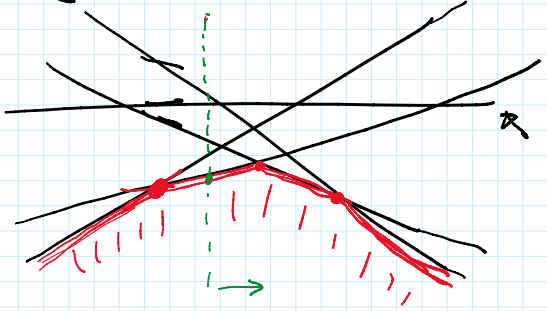
August 30, 2024 3:27 PM

Applications of Convex Hulls (& "Duality")

Problem 1 Given n lines $\ell_1, \dots, \ell_n$ in $\mathbb{R}^2$, compute their lower envelope (i.e. min of n linear fns in one var)



Def (Duality)

Given line ℓ with $y = mx + b$, define its dual pt $\ell^* = (m, -b)$

Given point $p = (\xi, \eta)$, define its dual line ℓ^* to have $y = \xi x - \eta$

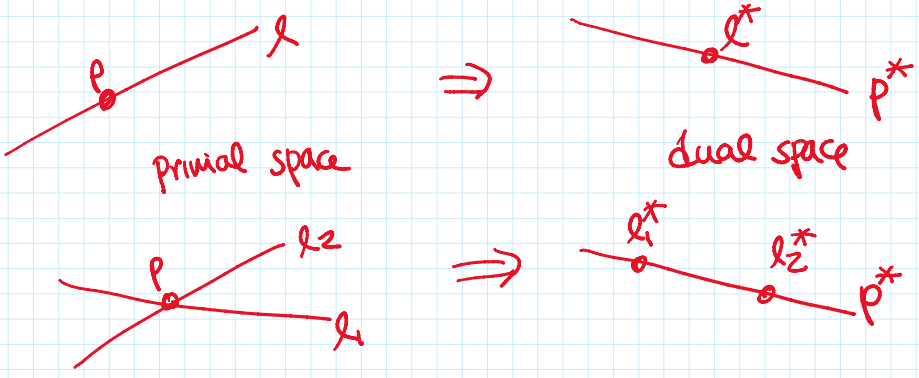
Obs p is on $\ell \iff \ell^*$ is on p^*.

Pf: $\eta = m\xi + b$ $\qquad -b = \xi m - \eta$

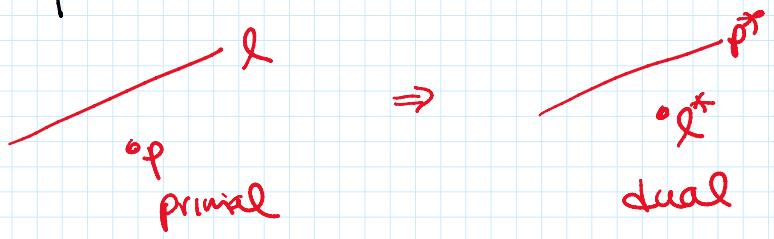
$\leq$ $\qquad \leq$ $\qquad$ □

Consequence: p is intersection of ℓ_1, ℓ_2

$\iff$ p^* is line thru ℓ_1^*, ℓ_2^*



Obs p is below $\ell \iff \ell^*$ is below p^*

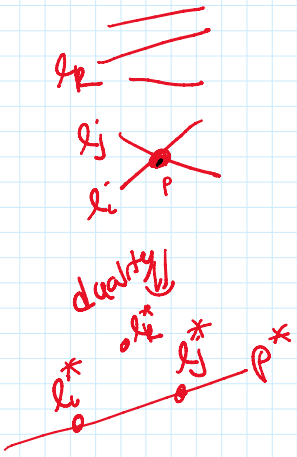


Consequence: $\ell_i \cap \ell_j$ is a vertex of lower envelope

$\iff$ $\ell_i \cap \ell_j$ below ℓ_k for all other k

$\iff$ ℓ_k^* above $\ell_i^* \ell_j^*$ for all k

$\iff$ $\ell_i^* \ell_j^*$ is an edge of the lower hull of $\ell_1^*, \ldots, \ell_n^*$

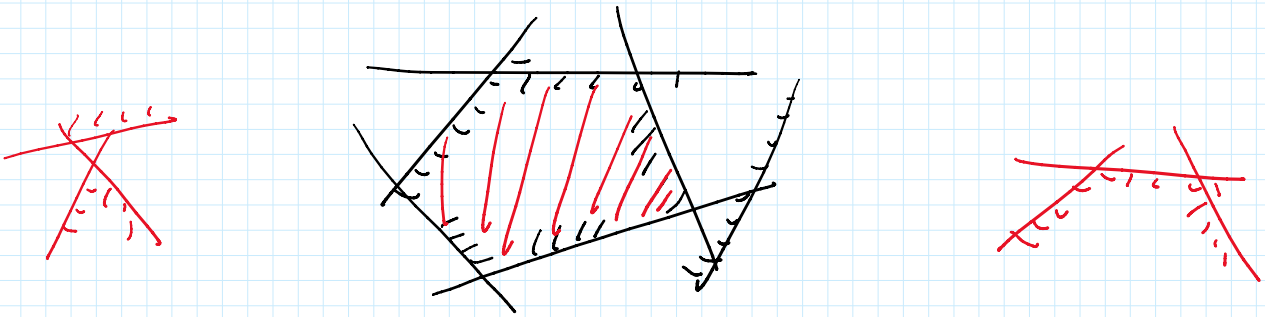


i.e. vertex of lower envelope $\iff$ edge of lower hull

edge ... vertex

$\Rightarrow$ $O(n \log n)$ time

Problem 2 Given n halfplanes $h_1, \ldots h_n$, containing the origin, in $\mathbb{R}^2$, compute their intersection



Def (Duality - alternate form)

Given halfplane h with inequality $ax + by \leq 1$, define dual pt $h^* = (a, b)$

Given halfplane h with inequality $ax + by \le 1$,
define dual pt $h^* = (a, b)$

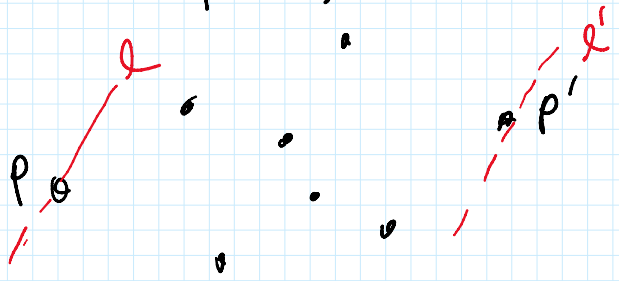
Given point $p = (\xi, \eta)$,
define dual halfplane p^* with inequality
$\xi x + \eta y \le 1$.

Obs $p \in h \iff h^* \in p^*$

$a\xi + b\eta \le 1 \qquad \xi a + \eta b \le 1.$

Consequence: halfplane intersection $\iff$ convex hull.

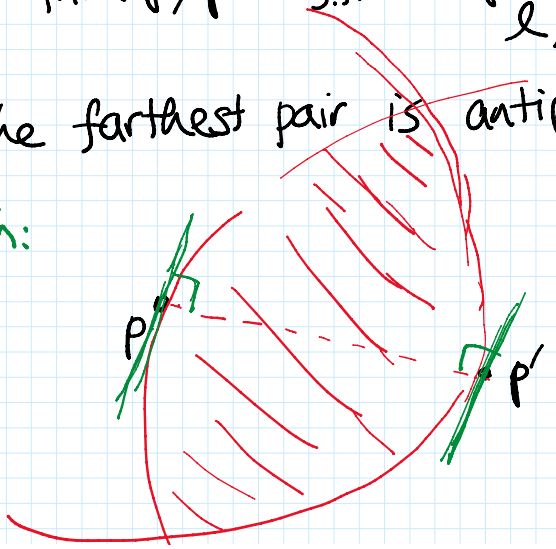
Problem 3 Given n pts in $\mathbb{R}^2$,
find farthest pair (diameter)



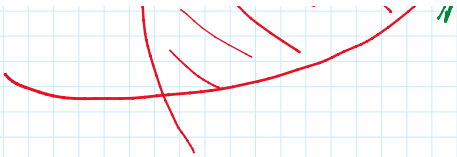
Def p, p' are antipodal iff $\exists$ parallel lines ℓ, ℓ' thru p, p' s.t. all pts are between ℓ, ℓ'.

Claim The farthest pair is antipodal.

Pf Sketch:

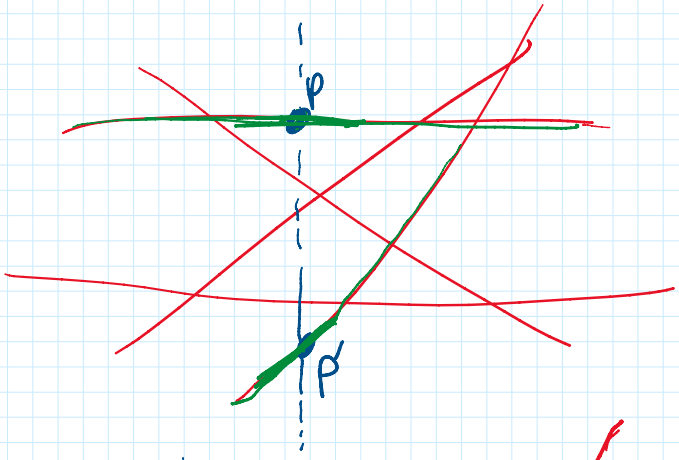


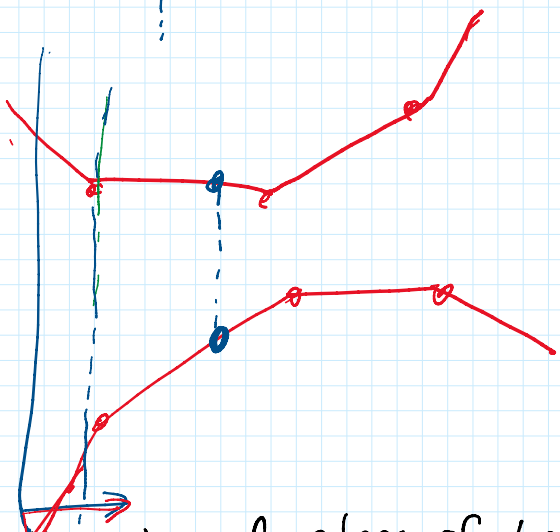
say pp' is farthest



New Problem find all antipodal pairs.

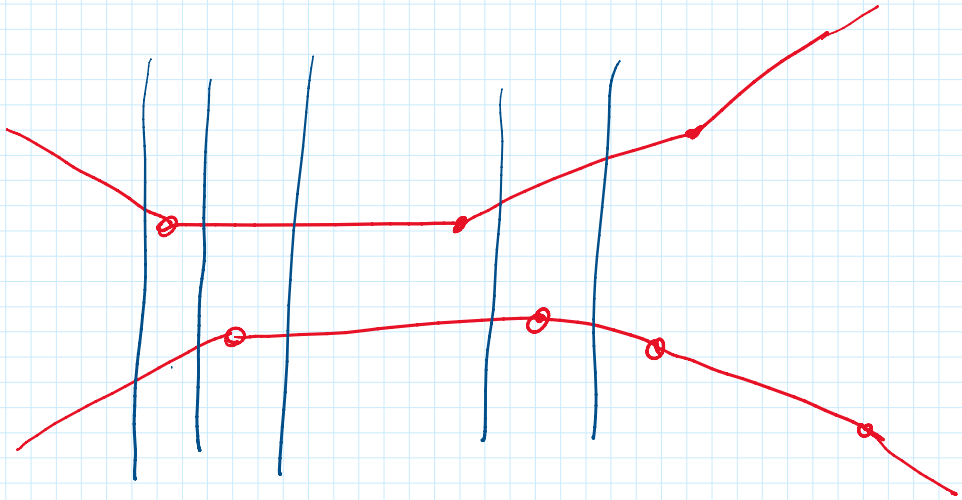
Dual: given n lines, find all pairs of lines (ℓ, ℓ') s.t. $\exists$ 2 points with same x-coord. p, p' with p on ℓ, p' on ℓ', s.t. all lines are between p, p'





i.e. find all pairs of edges of lower env & upper env. intersected by common vertical line

$\Rightarrow$ can be done by sweeping a vertical line from left to right



$\Rightarrow$ $O(n)$ time (in particular, $O(n)$ antipodal pairs)

total time: $\boxed{O(n \log n)}$