CS 498 TC, Fall 2024
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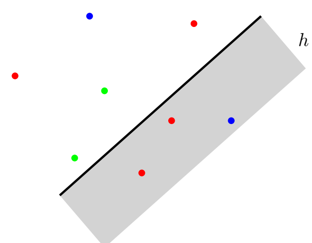
Homework 2 (due Oct 2 Wed 5pm)

Instructions: See previous homework.

1. [*34 pts*] In previous courses, you may have heard of the *graph coloring* problem: given an undirected graph, we want to assign colors to vertices so that no two adjacent vertices have the same color. The famous *4-color theorem* states that any *planar* graph can be colored with 4 colors.

   In this question, you will consider a geometric version of the coloring problem. Given a set P of n points, we want to assign colors to the points so that for every halfspace h that contains at least 2 points of P, not all points in $P \cap h$ have the same color.

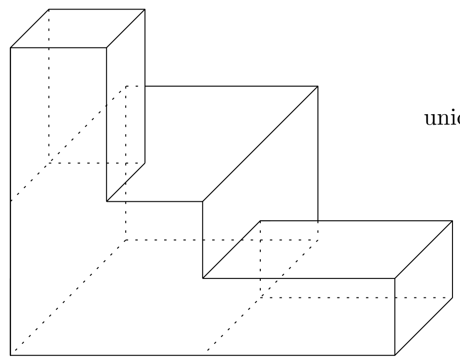
   (a) [*17 pts*] Show that for any point set P in 2D, it is always possible to color the points with 4 colors satisfying this condition. Also show that 4 is sometimes necessary (there exists a point set that needs 4).

   Hint: convex hull.

   (b) [*17 pts*] Show that for any point set P in 3D, it is always possible to color the points with c colors satisfying this condition, for some constant c.

   Hint: you may use the 4-coloring theorem for planar graphs stated above.

2. [*66 pts*] A *box* (or a *hyper-rectangle*) is the higher-dimensional generalization of a rectangle. In 3D, a box has 8 vertices, 12 edges, and 6 faces, where edges are parallel to the x-, y-, and z-axes. Let S be a set of n boxes in 3D, where each box has the origin as a vertex, i.e., each box is of the form $[0, a_i] \times [0, b_i] \times [0, c_i]$. The union U of the n boxes in S is a nonconvex polyhedron. We are interested in computing this polyhedron U.



union of 3 boxes

   (a) [*10 pts*] In 3D, show that U has $O(n)$ vertices, edges, and faces. Describe what a face of U in general looks like (it is a polygon satisfying what properties?).

   (b) [*16 pts*] Consider an *incremental* approach to computing U. Prove that if we randomize the order of insertion of the boxes, the expected total number of vertices, edges, and faces created and destroyed is $O(n)$.

   (c) [*30 pts*] Consider an incremental approach, but this time, consider inserting the boxes in decreasing order of z-coordinates c_i. Prove that with this insertion order, the total number of vertices, edges, and faces created and destroyed is always $O(n)$. Consequently, describe a deterministic $O(n \log n)$-time algorithm to compute the polyhedron U in 3D (using this incremental approach with appropriate data structures).

   Note: you can alternatively view the process as sweeping a plane in decreasing z-order.

   (d) [*10 pts*] If we do not assume that each box has the origin as a vertex, does U still have $O(n)$ vertices, edges, and faces? Explain.