# Chapter 16

# Motion Planning

By Sariel Har-Peled, March 7, 2023[1]

> 'I see Barsad, and Cly, Defarge, The Vengeance, the Juryman, the Judge, long ranks of the new oppressors who have risen on the destruction of the old, perishing by this retributive instrument, before it shall cease out of its present use. I see a beautiful city and a brilliant people' rising from this abyss, and, in their struggles to be truly free, in their triumphs and defeats, through long long years to come, I see the evil of this time and of the previous time of which this is the natural birth, gradually making expiation for itself and wearing out.
>
> – A Tale of Two Cities, Charles Dickens.

We cover here Chapter 13 in [BCKO08]

## 16.1. Work space and configuration space

The ***work space*** is the geometric environment where the robot moves. The ***configuration space*** is the space of configuration the robot might be in. For rotating and translating objects in $2d$, the configuration space is three dimensional (center location and orientation). A ***constraint*** for a robot is usually defined by the robot touching some obstacle, and this defines a "forbidden" surface the robot might not cross. Thus, given obstacles in the work space, in the configuration space we get various constraint surfaces (that might cross). The ***free space*** is the portion of the configuration space that encodes valid configurations for the robot. The motion planning problem asks to compute a path from a starting configuration in the free space to some other configuration.

### 16.1.1. Point robot

Given a set of disjoint polygons in the plane (i.e., obstacles), compute a path from two given locations $s$ to $t$ in the face space, for a point robot. This can be done using vertical decomposition in $O(n \log n)$ time, where $n$ is the total complexity of the polygonal obstacles. Specifically, we compute the vertical decomposition of the free space, and the dual graph – this is know as the ***road map***. A connectivity query is now no more than asking if the two trapezoids containing $s$ and $t$ are in the same connected component of the road map.

**Visibility graph and shortest path.** The following is somewhat more interesting.

**Lemma 16.1.1.** *Let $P$ be a set of disjoint polygons in the plane with total complexity $n$. Given points $s, t$ in the plane, one can compute the shortest path that avoids the obstacles in $O(n^2 \log n)$ time.*

*Proof:* Compute visibility graph, and then do Dijkstra. ■

## 16.2. Minkowski sum

Given two sets $P, Q \subseteq \mathbb{R}^d$, their ***Minkowski sum*** is

$$P \oplus Q = \{p + q \mid p \in P \text{ and } q \in Q\}.$$

We assume that we are dealing with a translating robot $R$ (i.e., a convex polygon) that has a reference point inside it. We denote by $R(x, y)$ the location of the robot if we place the reference point in $(x, y)$. Let $-R(0, 0)$ be the reflection of the robot through the origin.

**Lemma 16.2.1.** *Let $P$ be an obstacle, and let $R$ be a translating robot. The "forbidden" space induced by $P$ for $R$ is the set $P \oplus \Big(-R(0, 0)\Big)$.*

*Proof:* For a point $s \in \mathbb{R}^2$, if there is a point $q \in R(s) \cap P$, then $q - s \in R(0, 0) \iff s - q \in -R(0, 0)$. The later implies that $s = s - q + q \in -R(0, 0) \oplus P$.

As for the other direction, assume that $t \in -R(0, 0) \oplus P$. The there is a point $p \in P$, and $-r \in -R(0, 0)$, such that $t = p + (-r)$. That is $r \in R(0, 0)$, which implies that $p = t + r$, which implies that $p \in R(t)$. But that in turn implies that $p \in P \cap R(t)$. Namely, $t$ is a forbidden location. ■

**Lemma 16.2.2.** *Given two convex polygons of total complexity $n$ and $m$ respectively, one can compute their Minkowski sum in $O(n + m)$ time.*

Tow regions $X, Y$ in the plane are ***pseudodiscs*** if their boundaries intersect at most twice.

**Lemma 16.2.3.** *Let $P, Q$ be two disjoint convex polygons, and let $R$ be another convex polygon. Then $P \oplus R$ and $Q \oplus R$ are pseudodiscs.*

*Proof:* If not, $K_{3,3}$ can be drawn in the plane. ■

**Lemma 16.2.4.** *Given a set of polygons that are pseudodiscs with total complexity $n$, their union complexity is $O(n)$.*

**Lemma 16.2.5.** *For a transitional robot that is a convex polygon of constant complexity, and given a set of disjoint polygonal obstacles of total complexity $n$, the free space has complexity $O(n)$. This free space can be computed using divide-and-conquer in $O(n \log^2 n)$ time.*

## 16.3. Bibliographical notes

Our presentation more or less follows Chapter 13 in [BCKO08]

## References


[BCKO08] M. de Berg, O. Cheong, M. J. van Kreveld, and M. H. Overmars. *Computational Geometry: Algorithms and Applications*. 3rd. Santa Clara, CA, USA: Springer, 2008.