Chapter 15

Delaunay Triangulations

By Sariel Har-Peled, March 3, 2023[1]

"See? Genuine-sounding indignation. I programmed that myself. It's the first thing you need in a university environment: the ability to take offense at any slight, real or imagined."

Robert Sawyer, Factoring Humanity,

15.1. Delaunay triangulations

Given a set P of n points in the plane (in general position), a ***triangulation*** is a maximal planar graph having the points of P as vertices, and edges as segments. The outer face of a triangulation of P is the convex-hull of P, and all other faces are triangles.

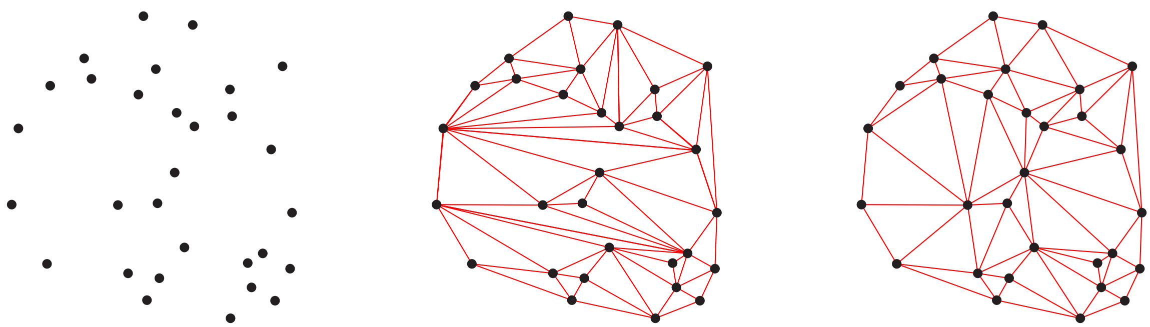

Figure 15.1.1: A point set, and two triangulations of the point set.

The following is an immediate consequence of Euler's formula.

Lemma 15.1.1. *Let P be a set of n points in the plane, and let k be the number of vertices of $\mathcal{CH}(P)$. Let Δ be any triangulation of P. Then Δ has $3n-3-k$ edges, and $2n-k-2$ (regular) triangles.*

Proof: Consider the planar graph G formed by adding a fake point to P outside its convex-hull and connecting it by (not necessarily) straight edges to the vertices of $\mathcal{CH}(P)$. This results in a triangulation of the plane. As proved already from Euler's formula, for a triangulation, we have $\mathsf{e}(\mathsf{G}) = 3\mathsf{v}(\mathsf{G}) - 6$. We thus have

$$\mathsf{e}(\Delta) = \mathsf{e}(\mathsf{G}) - k = 3\mathsf{v}(\mathsf{G}) - 6 - k = 3(n+1) - 6 - k = 3n - 3 - k.$$

Similarly, since $3\mathsf{f}(\mathsf{G}) = 2\mathsf{e}(\mathsf{G})$, we have

$$\mathsf{f}(\Delta) = \mathsf{f}(\mathsf{G}) - k + 1 = \frac{2}{3}\mathsf{e}(\mathsf{G}) - k + 1 = \frac{2}{3}(3n + 3 - 6) - k + 1 = 2n - 2 - k + 1 = 2n - k - 1.$$

Of course, the outer face is not counted as a regular triangle. ■

The ***circumcircle*** of three points p, q, s in the plane is the unique circle that passes through the three points. The center of the circumcircle is the unique point in the plane that is in equal distance to all three points. That is, the center is a vertex in the Voronoi diagram of the three sites.

Let m be the number of triangles in a triangulation $\boldsymbol{\Delta} = \boldsymbol{\Delta}(P)$. Consider the $3m$ angles in the triangles of $\boldsymbol{\Delta}$ and sort them by increasing order, and let $\sphericalangle\boldsymbol{\Delta} = (\alpha_1, \alpha_2, \ldots, \alpha_{3m})$ be the resulting vector, where $\alpha_i \leq \alpha_{i+1}$, for all i. The vector $\sphericalangle\boldsymbol{\Delta}$ is the ***signature*** of $\boldsymbol{\Delta}$. Given two triangulations $\boldsymbol{\Delta}$ and $\boldsymbol{\nabla}$ of P, let $\sphericalangle\boldsymbol{\Delta} = (\alpha_1, \ldots, \alpha_{3m})$ and $\sphericalangle\boldsymbol{\nabla} = (\beta_1, \ldots, \beta_{3m})$ be their corresponding signatures. We use $\succ$ to denote the lexicographical order on the signatures – that is, $\sphericalangle\boldsymbol{\Delta} \succ \sphericalangle\boldsymbol{\nabla} \iff \sphericalangle\boldsymbol{\Delta}$ is lexicographically larger than $\sphericalangle\boldsymbol{\nabla}$. Specifically, there is an index k, such that

$$\alpha_i = \beta_i, \text{ for } i = 1, \ldots, k-1, \text{ and } \alpha_k > \beta_k.$$

Given two points p, q the locus of all points seeing the two points in a certain fixed angle α is the union of two circular arcs. To see this, consider a point s seeing the p and q in angle α, and recall that the center of the circumcircle sees the two points in angle 2α (this follows from elementary geometric arguments). It is not hard to see that any point in this circular arc sees the point in the same angle. See Figure 15.1.2. The interior of the region formed by these two circular arcs is the set of all points that sees p, q with angle strictly larger than α. All points outside this region sees the pair of points with angle strictly smaller than α.

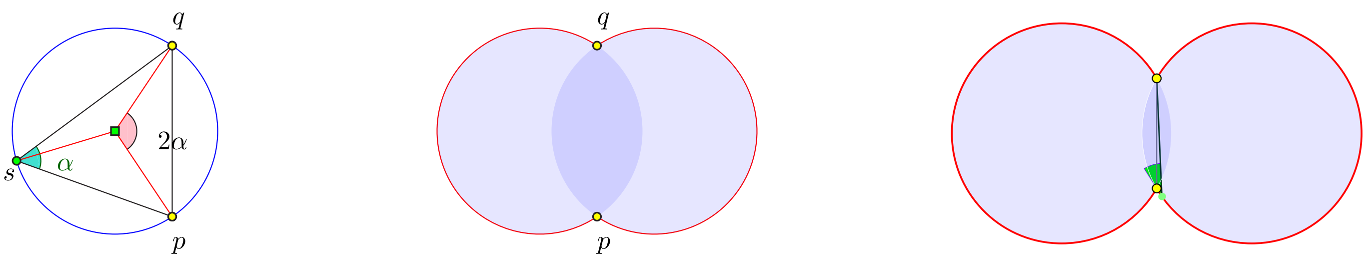

Figure 15.1.2: The circle supporting the points that see p, q with angle α. On the right is the region of all the points that sees p and q with angle $\geq \alpha$.

To get a handle of this entity, it is useful to consider the set of circles that have p and q on its boundary. Such a set of circles is a ***pencil***. See Figure 15.1.3 for an illustration. A pencil is a one dimensional family of shapes parameterized by the location of the center of the disk along the bisector line. One can continuously move between two circles in the pencil by continuously moving the center from the starting circle center to the target circle center.

An edge s in a triangulation is ***illegal*** if two triangles adjacent to it form a convex quadrilateral, and furthermore, the circumcircle of the one of the triangles contains the other vertex of the quadrilateral in it. So consider a convex quadrilateral with points $psqt$, such that circumcircle of $\triangle psq$ contains t in its interior. By continuous pencil argument, it is easy to see that the circumcircle of $\triangle ptq$ contains

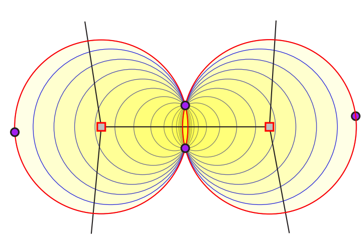

Figure 15.1.3: The portion of the pencil defined by circles that lie on a Voronoi edge.

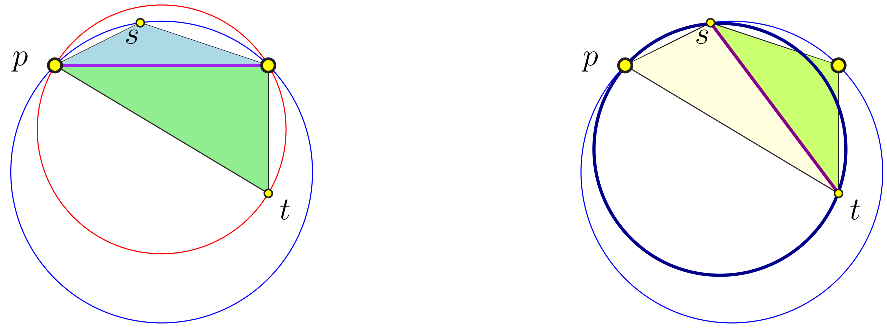

Figure 15.1.4: Illegal edge.

s in its interior. Thus, the above definition of ***illegal*** edge is consistent – if one triangle testifies the edge is illegal, so does the adjacent triangle. See Figure 15.1.4. To get rid of an illegal edge, instead of deporting it, we ***flip*** the illegal diagonal of the quadrilateral, replacing it by the other quadrilateral. It is easy to verify that the new diagonal is now legal. Furthermore, the smallest angle in the two new triangles is bigger than it was before. See Figure 15.1.4.

Lemma 15.1.2. *Let $P = \{p, s, q, t\}$ be a set of four points in the plane in convex position, and let $\boldsymbol{\Delta}$ be the triangulation of P having pq as a diagonal, which is illegal. Let $\boldsymbol{\nabla}$ be the triangulation of P that has st as a diagonal. Then st is a legal diagonal, and furthermore, $\measuredangle\boldsymbol{\nabla} \succ \measuredangle\boldsymbol{\Delta}$.*

Corollary 15.1.3. *Let P be a set of n points in the plane, and let $\boldsymbol{\Delta}$ be a triangulation. Let $\boldsymbol{\nabla}$ be the result of flipping an illegal diagonal in $\boldsymbol{\Delta}$. We have that $\measuredangle\boldsymbol{\nabla} \succ \measuredangle\boldsymbol{\Delta}$.*

Proof: Follows readily from the above lemma, as the flip effects only the angles involved in the flip. ∎

A ***legal triangulation*** is a triangulation that all its edges are legal. Observe that the set of triangulation is finite, and have an ordering on the signatures, which strictly increases after every flip. It follows that the greedy algorithm that starts with arbitrary triangulation and repeatedly flips illegal edges, ends up with a legal triangulation.

Lemma 15.1.4. *Let $\boldsymbol{\Delta}$ be a legal triangulation of P. Then, for all triangles $\triangle \in \boldsymbol{\Delta}$, their associated circumcircle contains no points of P in its interior.*

Proof: Assume for contradiction that this is false, and let $\triangle \in \boldsymbol{\Delta}$ be the offending triangle, with $p \in P$ being the point lying in the interior of its circumcircle. Set $\triangle_1 = \triangle$. In the following, let C_i denote the circumcircle of $\triangle_i$.

In the ith iteration, for $i \geq 1$, we have that p is contained in the interior of C_i, and must be separated from the interior of $\triangle_i$ by one of its edges, say e_i. Let $\triangle_{i+1}$ be the triangle adjacent to $\triangle_i$ through e_i in $\boldsymbol{\Delta}$. If p is a vertex of $\triangle_{i+1}$ then we found an illegal edge (i.e., e_i), which is a contradiction to $\boldsymbol{\Delta}$ being a legal triangulation. See Figure 15.1.5.

Otherwise, observe that $\triangle_i$ and $\triangle_{i+1}$ both belong to the pencil defined by the endpoints of e_i. In particular, C_{i+1}, the circumcircle of $\triangle_{i+1}$, fully contains the portion of C_i that is on the same side of e_i as p. That is, p is contained in C_{i+1}. We now continue the argument to the next iteration – arguing about $\triangle_{i+1}, C_{i+1}$ and p. We claim that this "walk" in the triangulation can not cycle – indeed, we replaced $\triangle_i$ by a triangle $\triangle_{i+1}$ that is strictly closer to p (i.e., $\mathsf{d}(p, \triangle_i) = \min_{q \in \triangle_i} \|pq\|$ is strictly decreasing as a function of i). Since the triangulation $\boldsymbol{\Delta}$ is finite, this process must stop, which implies that there is an illegal edge in the triangulation $\boldsymbol{\Delta}$. Ludicrous! A contradiction. ∎

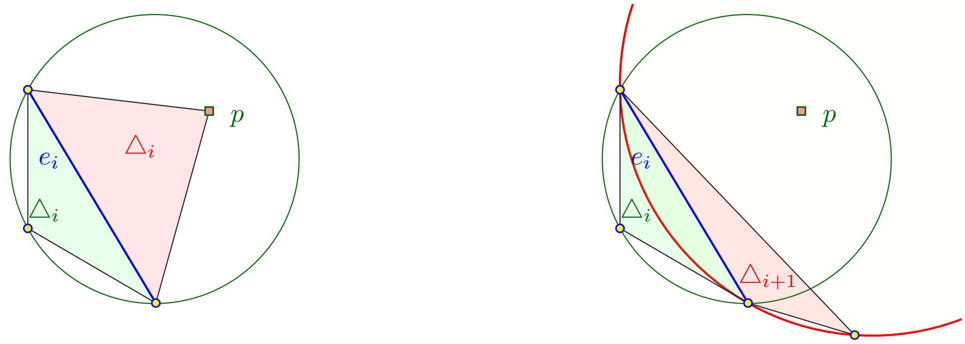

Figure 15.1.5: Walking like an Egyptian in a Delaunay triangulation.

Lemma 15.1.5. *Assuming general position on P, and that $|P| > 2$, consider a legal triangulation Δ of P, and consider an edge $pq \in \Delta$. Then, there is circle C that contains only p, q on its boundary, and no points of P in its interior.*

Proof: The case that pq is an edge of the convex-hull of P is immediate. Otherwise, there are two triangles $\triangle, \triangle' \in \Delta$ that share the edge pq. Clearly, the circumcircles C, C' of these two triangles belong to the pencil of p, q. Pick any circle C'' in between C and C' in this pencil. Clearly, C'' is the desired circle. ■

Lemma 15.1.6. *Assuming general position, the legal triangulation of P is unique.*

Proof: Assume for contradiction that there are two different legal triangulations Δ and Δ' of P. Then there must be an edge $e \in \Delta$ and an edge $e' \in \Delta'$ such that they intersect in their interior (as otherwise, the two triangulations are the same). By Lemma 15.1.5, there is a circle C (resp. C') that have the endpoints of e (resp. e') on its boundary, and no other point of P either on its boundary or its interior.

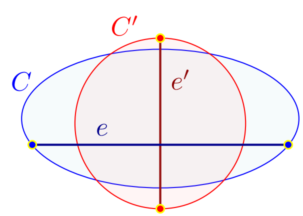

Figure 15.1.6

The situation is depicted in Figure 15.1.6 – we have two distinct circles that supports two distinct segments that intersect. This in turn implies that these two circles must have four intersections, which is impossible. A contradiction. ■

This unique legal triangulation is the ***Delaunay triangulation*** of P, and let $\mathcal{D}(P)$ denote this triangulation.

Lemma 15.1.7. *The dual graph of the Delaunay triangulation Δ of P is the Voronoi diagram of P.*

Proof: The dual graph of the Voronoi diagram is clearly a triangulation, as all vertices in the Voronoi diagram have degree three (assuming general position). An edge e of the Voronoi diagram encodes a pencil of circles passing through the two sites defining the edge. The two Voronoi vertices of this Voronoi edge corresponds in the dual to two triangles that form a quadrilateral, and the dual edge $e^{\star}$ is legal. Namely, the dual triangulation to the Voronoi diagram is the unique legal triangulation of P. ■

Definition 15.1.8. The triangulation Δ of P such that all circumcircle associated with a triangle of Δ contains no other points of P is well-defined and unique, and is the ***Delaunay triangulation*** of P.

Consider the triangulation Δ of P that maximizes the signature. Observe that Δ must be legal, as otherwise we can flip and increase the angles vector. It follows that the triangulation maximizing the angles vector is the Delaunay triangulation.

Corollary 15.1.9. *For a set P of points, the Delaunay triangulation of P maximizes the minimum angle among all triangulations of P.*

Thus, in this sense, the Delaunay triangulation is the one yields the "fattest" possible triangles. Another important consequence of the duality of the Voronoi diagram/Delaunay triangulation is the following characteristic.

The above implies the following much more convenient characterization of Delaunay triangulation – the original definition is cumbersome. One can of course start from the following as the definition of Delaunay triangulation, but then one needs to prove that it indeed from a triangulation, which is effectively what we did above.

Lemma 15.1.10. *Let P be a set of points in the plane. The points pq form an edge in the Delaunay triangulation $\mathcal{D}(P) \iff$ there exists disk that contains p, q, and no other point of P.*

Similarly, $\triangle pqs$ is a triangle in $\mathcal{D}(P) \iff$ the circumcircle of $p, q.s$ contains no other points of P in it.

15.2. Computing Delaunay triangulation via lifting

Consider the paraboloid $z = x^2 + y^2$ and a hyperplane $h \equiv z = \alpha x + \beta y + \gamma$. The intersection of these two surfaces is the set of points complying with the equation

$$
\begin{aligned}
x^2 + y^2 = \alpha x + \beta y + \gamma \iff \quad & \left(x^2 - \alpha x + \frac{\alpha^2}{4}\right) + \left(y^2 - \beta x + \frac{\beta^2}{4}\right) = \frac{\alpha^2}{4} + \frac{\beta^2}{4} + \gamma \\
\iff \quad & \left(x - \frac{\alpha}{2}\right)^2 + \left(y - \frac{\beta}{2}\right)^2 = \frac{\alpha^2}{4} + \frac{\beta^2}{4} + \gamma.
\end{aligned}
$$

Namely, the projection of the intersection to the xy-plane is a circle centered at $(\alpha/2, \beta/2)$ and is of radius $\sqrt{\frac{\alpha^2}{4} + \frac{\beta^2}{4} + \gamma}$. Clearly, the points inside the disk are below h when lifted to the paraboloid, and the points outside are above the hyperplane.

Note that this mapping is bijective – given any circle in the plane, we can compute the plane in three dimensions that its intersection with the paraboloid when projected to the xy-plane is this circle.

So, consider a point set $\mathsf{P} \subseteq \mathbb{R}^2$. Let $\mathsf{Q} = \left\{ (x, y, x^2 + y^2) \;\middle|\; (x, y) \right\}$ be the point set lifted to the paraboloid. Clearly, any disk that has three points of $q, \mathsf{s}, t \in \mathsf{P}$ on its boundary, corresponds to a plane that passes through the corresponding three points of Q, the projection of the intersection of this plane to the xy plane is the boundary circle of this disk, and all the other points of Q are above this plane. In particular, the Delaunay triangle $\triangle \mathsf{q}\mathsf{s}t$ corresponds to a face of the convex-hull of Q that has the three lifted points as its vertices. We thus proved the following amazing result (we argued about it in two dimensions, but the argumentation holds in any dimension).

Lemma 15.2.1. *The Delaunay triangulation of a set of points P in $\mathbb{R}^{\mathsf{d}}$, can be computed by:*

(A) *Lifting the points of* P *to the paraboloid in* $\mathbb{R}^{\mathsf{d}+1}$*, resulting in a point set* Q*.*
(B) *Computing the convex hull of* Q*.*
(C) *Projecting down the "down" faces of the convex-hull results in the Delaunay triangulation of* P*.*

What about flipping? The current triangulation can be interpreted as a surface in 3d. As long as this surface is not convex, we can locally perform a flip – this is equivalent to two replacing two adjacent triangles, by the two bottom triangles that form the convex hull of the four points in three dimensions. Visually, we are filling a cavity in the surface by filling in a simple. We can repeat this process till we arrive to the convex-hull.

Since we already seen how to compute the convex-hull of n points in 3d in $O(n \log n)$ time, we readily get the following.

Theorem 15.2.2. *The Delaunay triangulation of* n *points in 3d can be computed in* $O(n \log n)$ *(expected) time.*

15.3. Bibliographical notes

Our presentation more or less follows Chapter 9 in [BCKO08]

References

[BCKO08] M. de Berg, O. Cheong, M. J. van Kreveld, and M. H. Overmars. *Computational Geometry: Algorithms and Applications*. 3rd. Santa Clara, CA, USA: Springer, 2008.