Chapter 14

Voronoi diagrams

By Sariel Har-Peled, February 28, 2023[①]

They amputated your thighs of my hips.
As far as I'm concerned they are all surgeons.
All of them.

They dismantled us each from the other.
As far as I'm concerned they are all engineers.
All of them.

A pity. We were such a good and loving invention.
An airplane made from a man and a wife.
Wings and everything.
We hovered a little above the earth.

We even flew a little.

Yehuda Amichai, A pity. We were such a good invention., Philip Roth

14.1. Voronoi diagrams

The *Voronoi diagram* of a point set P is a partition of the plane into regions, where each point of P servers as the nearest neighbor to its region. Its dual structure, the *Delaunay triangulation* provides a canonical triangulation of points with many desirable properties.

14.1.1. Definition

Given a set P of n points in the plane, consider the assignment of each point of the plane to its nearest neighbor in P. Formally, the ***distance*** of a point $p \in \mathbb{R}^2$ from P is

$$\mathsf{d}(p, \mathsf{P}) = \min_{q \in \mathsf{P}} \|pq\| .$$

The ***Voronoi cell*** of a site $p \in \mathsf{P}$ is the set of all points for which p is their nearest neighbor in P, that is

$$V_p = \left\{ q \in \mathbb{R}^2 \;\middle|\; \|qp\| = \mathsf{d}(q, \mathsf{P}) \right\} .$$

Thus, the cells induced by the points of P partitions the plane into the ***Voronoi diagram*** of P.

For our purposes, it would be useful to consider the labeling function

$$V(q) = \arg\min_{p \in \mathsf{P}} \|qp\| .$$

Namely, the Voronoi diagram labels the points in the plane with their nearest neighbor in P, where the Voronoi cell of a site $p \in \mathsf{P}$, is the set of all points labeled by p.

Examples. The ***bisector*** of two points p, q is a line, denoted by $p|q$, that is orthogonal to the segment pq, and passes through the middle point of this segment. Thus, the Voronoi diagram of two points is the partition of the plane formed by this bisector. See Figure 14.1.1.

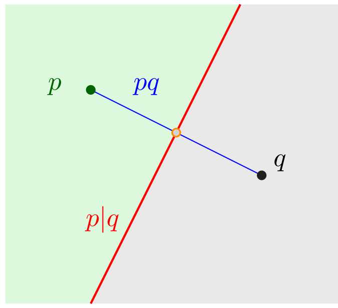

Figure 14.1.1: Bisector line, and the Voronoi diagram of two points.

For three points p, q, s, their Voronoi partition is formed by three rays emanating from a single point. This is illustrated in Figure 14.1.2. Indeed, consider the ***circumcircle*** of these three points – we remind the reader that this is the unique circle that passes through the three points. The center of this circle $\overline{\mathsf{c}}$ is in equal distance to all three points, which implies that $\overline{\mathsf{c}} \in p|q$, $\overline{\mathsf{c}} \in p|\mathsf{s}$, and $\overline{\mathsf{c}} \in q|\mathsf{s}$. The point $\overline{\mathsf{c}}$ is the ***Voronoi vertex*** of p, q, and s. Note, that the Voronoi vertex $\overline{\mathsf{c}}$ might lie outside the triangle $\triangle pq\mathsf{s}$.

The easiest way to see what this

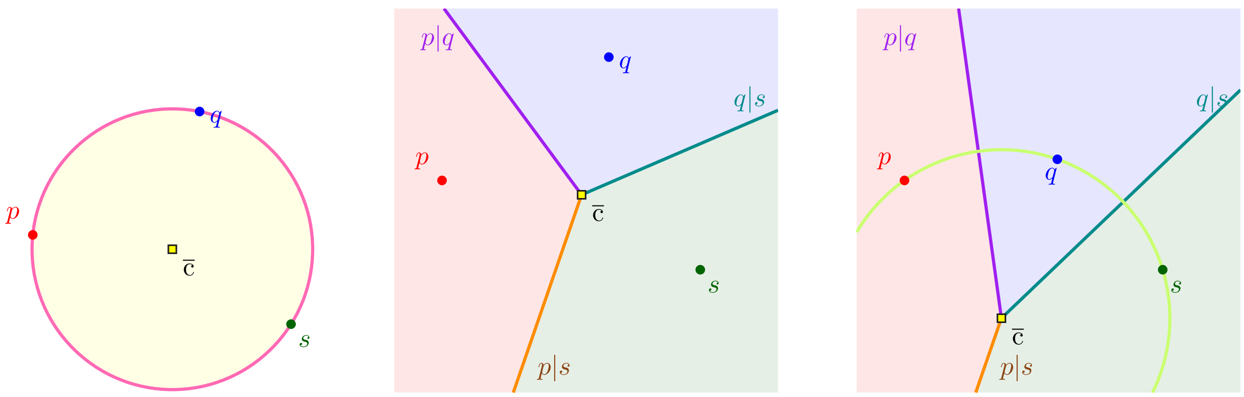

Figure 14.1.2: Voronoi diagram of three points.

Points and cones. Let $\mathsf{P} = \{p_i1, \ldots, p_in\}$ be a set of n points in the plane, where $p_ii = (x_i, y_i)$, for $i = 1, \ldots, n$. For each point, we define the "natural" distance function

$$f_i(x, y) = \|(x, y)p_ii\| = \sqrt{(x - x_i)^2 + (y - y_i)^2}.$$

The graph of the function $f_i(x, y)$ is a cone in three dimensions:

(A) Its apex is located at $(p_ii, 0) = (x_i, y_i, 0)$.
(B) Its center of symmetry is a vertical ray emanating from $(p_ii, 0)$.
(C) Its opening angle is 45 degrees.

In particular, the Voronoi diagram of P is the labeling of the plane by the function:

$$V(x, y) = \arg\min_i \|(x, y)p_i\| = \arg\min_i f_i(x, y).$$

14.1.2. Lifting and duality for Voronoi diagrams

We are interested in computing the minimization diagram $F(x, y) = \min_i f_i(x, y)$. Namely, for any $\mathsf{q} = (x, y) \in \mathbb{R}^2$, the point q is in the Voronoi diagram of p_ii if and only if $i = \arg\min_k f_k(x, y)$. In particular, for any mapping $g : \mathbb{R}^+ \to \mathbb{R}^+$ that is strictly monotone, we have that

$$\arg\min_k f_k(x, y) = \arg\min_k g(f_k(x, y)).$$

As such, taking $g(x) = x^2$, which is strictly monotone for non-negative x, we have that

$$\begin{aligned}
\arg\min_k f_k(x, y) &= \arg\min_k (f_k(x, y))^2 = \arg\min_k \left(\sqrt{(x - x_k)^2 + (y - y_k)^2}\right)^2 \\
&= \arg\min_k \left(x^2 - 2xx_k - x_k^2 + y^2 - 2yy_k - y_k^2\right) \\
&= \arg\min_k \left(\left(x^2 + y^2\right) - 2xx_k - x_k^2 - 2yy_k - y_k^2\right) \\
&= \arg\min_k \left(-2xx_k - x_k^2 - 2yy_k - y_k^2\right),
\end{aligned}$$

the last follows as subtracting the same quantity from all the functions, preserve the function that achieves the minimum. Note, that the resulting functions are now hyperplanes in $\mathbb{R}^3$, and we want to compute their lower envelope.

The above argument also extends to higher dimensions, and we conclude the following.

Lemma 14.1.1. *Computing the Voronoi diagram of n points in $\mathbb{R}^{\mathsf{d}}$, can be done by computing the lower envelope of n hyperplanes in $\mathbb{R}^{\mathsf{d}+1}$.*

Thus, given a set P of points in $\mathbb{R}^{\mathsf{d}}$, we get a set of hyperplanes $\mathcal{H}$ in $\mathbb{R}^{\mathsf{d}+1}$. By duality, computing the lower envelope of $\mathcal{H}$ is equivalent to computing the upper portion of the convex-hull of $\mathsf{Q} = \mathcal{H}^\star$.

Corollary 14.1.2. *Given a set P of n points in 2d, one can compute their Voronoi diagram in $O(n \log n)$ (expected) time.*

Proof: By lifting and duality, this is simply computing the upper part of the convex-hull of n points in 3d, which can be done in $O(n \log n)$ expected time. ■

14.2. Other diagrams

14.2.1. Power diagrams

14.2.2. Additive Voronoi diagrams

14.2.3. Voronoi diagrams of convex regions

14.3. Bibliographical notes