Chapter 2

Arrangements, Vertical Decomposition, and Point Location

By Sariel Har-Peled, January 26, 2023[1]

The Party told you to reject the evidence of your eyes and ears. It was their final, most essential command.
– 1984, George Orwell.

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2.1. Arrangements

Given a set of segments S, they decompose the plane into regions – indeed, imagine the plane being a large piece of paper, and imagine cutting it along the segments of S. The paper breaks into several pieces. Every piece is a face. Formally, a ***face*** in the arrangement of $\mathcal{A}(S)$ is a maximal connected subset of

$$\mathbb{R}^2 \setminus \cup S = \mathbb{R}^2 \setminus \bigcup_{s \in S} s.$$

A ***vertex*** of $\mathcal{A}(S)$ is either an endpoint of a segment of S, or an intersection point of two segments of S. Let $\mathsf{V} = \mathsf{V}(\mathcal{A}(S))$ be the set of all vertices of $\mathcal{A}(S)$. Removing the vertices of V from the segments of S form the edges. Formally, an ***edge*** of $\mathcal{A}(S)$ is a maximum connected component of

$$(\cup S) \setminus \mathsf{V}.$$

Let $\mathsf{E} = \mathsf{E}(\mathcal{A}(S))$ denote the set of all edges of $\mathcal{A}(S)$. The arrangement of S form an embedding of the planar graph formed by V and E. Indeed, the vertices of the graph are V, and the edges are E. For every edge in this graph, we have an explicit information of how to draw it in the plane – as a straight segment. This drawing is ***planar*** – no two edges cross in their interior, and no edge has an endpoint in the interior of another edge. A graph is ***planar*** if it has such a drawing. An ***embedding*** is a specific drawing of a planar graph.

2.1.1. Representing arrangements using doubly-connected edge list

2.1.2. Overlaying arrangements using sweeping

2.1.3. Vertical decomposition and point-location