Chapter 2

Introduction to Planar Graphs

By Sariel Har-Peled, January 31, 2023[1]

> The Party told you to reject the evidence of your eyes and ears. It was their final, most essential command.
> – 1984, George Orwell.

Version: 0.1

2.1. Planar graphs

A graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ is ***planar*** if it can be drawn in the plane. In a ***drawing*** of a graph, a vertex is mapped to a point, and an edge is a simple curve – specifically, a curve is the image of a continuous one-to-one mapping from $[0, 1]$ to the plane, where the two endpoints are vertices. This definition of a curve is somewhat informal (for example, it includes space-filling curves), but formalizing it requires opening a Pandora box, so lets for the time remain with this definition with the understanding that we restrict ourselves to well-behaved curves (whatever that means formally).

As such, in a valid planar drawing, not two curves (i.e., edges) can intersect in their interior. Thus, intuitively, being a planar graph – that is, having a planar drawing – is a restrictive property. It is not hard to convince oneself, for example, that it is not possible to draw K_5 and $K_{3,3}$. See Figure 2.1 for example.

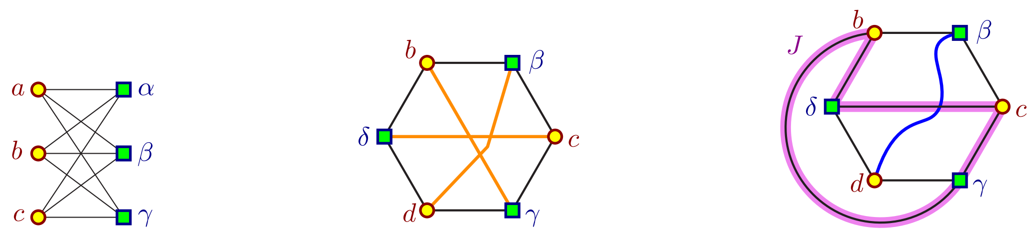

Figure 2.1: Attempts to draw $K_{3,3}$ in the plane.

Given a drawing of a planar graph, if we cut the plane along the edges of the graph (i.e., the curves formed by the edges), we break the plane into connected pieces. Each such connected piece is a ***face***. Thus, a planar graph G, has vertices, edges, and faces.

A graph is ***simple*** if it has no parallel edges (i.e., two edges or more with the same endpoints).

Theorem 2.1.1 (Euler's formula). *For a simple connected planar graph* G*, let* v*,* e *and* f *denote the number of vertices, edges, and faces of* G*. We have that* $f - e + v = 2$*.*

Proof: (Informal.) Let T be a spanning tree of G. Clearly, in the drawing of G, if we draw only T, it has $n-1$ edges, one face, and n vertices. As such, $f(T) - e(T) + v(T) = 1 - (n-1) + n = 2$, as claimed. Now, order the edges of $\mathsf{G} - T$ in an arbitrary order, and add them one by one to the drawing. Each edge added increases the number of edges by one, and the number of faces by one. Thus implying the claim. (The more formal way to write this proof is by induction on the number of edges.) ∎

An easy way to remember Euler's formal is that it alternates the signs between the entities it sums (i.e., vertices, edges, faces), with the alternation done on the dimension. As for the constant, just draw a triangle – it has $f = 2$, $e = 3$, and $v = 3$, which implies that the constant is 2, as desired.

A ***closed Jordan curve*** is a closed curve, that does not self intersect, that can be continuously deformed into a circle. The following theorem is deceptively obvious, and has interesting history – it is the main tool when arguing about planarity.

Theorem 2.1.2 (Jordan curve theorem). *A closed Jordan curve J partition the plane into two open connected components – the interior and the exterior. Any curve connecting a point in the interior, to a point in the exterior must intersect J.*

A natural process when given a drawing of a simple planar graph G, is to try and add edges (i.e., curves to it) till no edge can be added. Such a drawing results in a ***triangulation*** – as every face in the drawing has exactly three edges. As such, for this "enriched" planar graph H, we have that $2\mathsf{e} = 3\mathsf{f}$, where $\mathsf{e} = |\mathsf{E}(\mathsf{H})|$, $\mathsf{f} = |\mathsf{F}(\mathsf{H})|$, and $\mathsf{F}(\mathsf{H})$ denote the faces of H. For $\mathsf{v} = \mathsf{v}(\mathsf{H}) = |\mathsf{V}(\mathsf{H})| = |\mathsf{V}(\mathsf{G})|$, Euler's formula now implies that

$$2 = \mathsf{f} - \mathsf{e} + \mathsf{v} = (2/3)\mathsf{e} - \mathsf{e} + \mathsf{v} \qquad \Longrightarrow \qquad \mathsf{e} = 3\mathsf{v} - 6.$$

Since $\mathsf{e}(\mathsf{G}) \leq \mathsf{e}(\mathsf{H})$, we get the following.

Lemma 2.1.3. *In a (simple) planar graph G with v vertices, there are at most $3\mathsf{v} - 6$ edges.*

2.1.1. Degeneracy and colorability

Definition 2.1.4. *Consider a graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$. A* **subgraph** *$\mathsf{H}$ of G is a graph such that $\mathsf{V}(\mathsf{H}) \subseteq \mathsf{V}$ and $\mathsf{E}(\mathsf{H}) \subseteq \mathsf{E}$.*

That is, a subgraph of a graph is what you get from deleting some edges and vertices from the original graph.

Definition 2.1.5. *Consider a graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$. For a set $X \subseteq \mathsf{V}$, its* **induced subgraph** *is $\mathsf{G}_X = \bigl(X, \{uv \mid uv \in \mathsf{E} \text{ and } u, v \in X\}\bigr)$*

Clearly, a subgraph (or induced subgraph) of a planar graph is planar.

Definition 2.1.6. *A graph G is* **k-degenerate** *if it has a vertex u of degree at most k, and $\mathsf{G} - u$ is k-degenerate.*

A stronger property that implies that a graph G is k-degenerate is that for any induced subgraph H of G, we have the property that H has a vertex of degree k.

Lemma 2.1.7. *Planar graphs are 5-degenerate.*

Proof: Let G be a planar graph, and let H be any induced subgraph of G. Let $\mathsf{v} = \mathsf{v}(\mathsf{H})$. There is a vertex u in H of degree $2\mathsf{e}(\mathsf{H})/\mathsf{v} \leq 2(3\mathsf{v} - 6)/\mathsf{v} < 6$. Namely H has a vertex of degree at most 5, which implies the claim. ■

Lemma 2.1.8. *A k-degenerate graph is $(k+1)$-colorable. As such, planar graphs are 6-colorable.*

Proof: The proof is by induction. The claim is obvious if the graph G has $\mathsf{v} \leq k+1$. Otherwise, G is k-degenerate, and let u be any of its vertices of degree at most k. The graph $\mathsf{G} - u$ is k-degenerate (since this is a hereditary property), and recursively color it using $k+1$ colors. Now, we put u back – it has at most k neighbors, which implies that out of our palette of $k+1$ colors, one of them is not used by the neighbors of u. We set the color of u to be this color.

Since planar graphs are 5-degenerate the claim follows. ■

Famously, planar graphs can be colored using only four colors (i.e., the four-color theorem). This also implies that K_5 is not planar, since it has $\mathsf{v} = 5$, and $\mathsf{e} = \binom{5}{2} = 10$ edges, but $10 = \mathsf{e} \not\leq 3\mathsf{v} - 6 = 9$.

2.1.2. Drawing planar graphs and separators

Given an a graph without a drawing, a natural task is to decide if it is planar. That is, can it be drawn in the plane. This can be done in linear time using somewhat involved algorithms that we are not going to describe here. Maybe the most algorithmically useful property of planar graphs is that they have a separator.

Theorem 2.1.9. *Let G be a planar graph with v vertices. Then, one can compute a set $Z \subseteq \mathsf{V}(\mathsf{G})$ of $O(\sqrt{n})$ vertices, such that $\mathsf{G} - Z$ is disconnected, and every connected component has at most $(2/3)n$ vertices. The set Z is a* **separator** *of G, and it can be computed in linear time.*

2.1.3. Kuratowski's and Wagner's theorems

We state, without proof, the following beautiful characterization of planar graphs. A ***subdivision*** of a graph is formed by subdividing its edges into paths of one or more edges. A graph H ***contains a subdivision*** of G, if there is a subgraph of $\mathsf{H} \subseteq \mathsf{H}$, that is a subdivision of H – formally, H is ***isomorphic*** to some subdivision of G. Here, two graphs G_1 and G_2 are ***isomorphic*** if up to renaming of vertices, they are the same graph.

Theorem 2.1.10 (Kuratowski's theorem). *A graph G is planar if and only if it does not contain $K_{3,3}$ or K_5 as a subdivision.*

A closely related and equivalent result is Wagner's theorem, but before stating it we need a few definitions. For a graph G, and an edge e, the ***contraction*** of e results in the graph G/e, which is the result of merging the two vertices that are the endpoint of e into a single vertex. This might result in self loops, that we remove, and parallel edges, which we merge into a single edge.

It is easy to verify that edge contraction, deletion, and vertex deletion, are operations that preserve planarity – that is a planar graph after you apply these operation to them remain planar.

A graph H is a ***minor*** of G if there is a sequence of edge deletions, vertex deletions, and edge contractions that transform G to H. Intuitively, having, say $K_{3,3,}$, as a minor in a graph G implies that the $K_{3,3}$ is "hiding" somewhere inside G. Not surprisingly, both $K_{3,3}$ and K_5 can not hide in a planar graph.

Theorem 2.1.11 (Wagner's theorem). *A graph G is planar if and only if it does not contain $K_{3,3}$ or K_5 as a minor.*

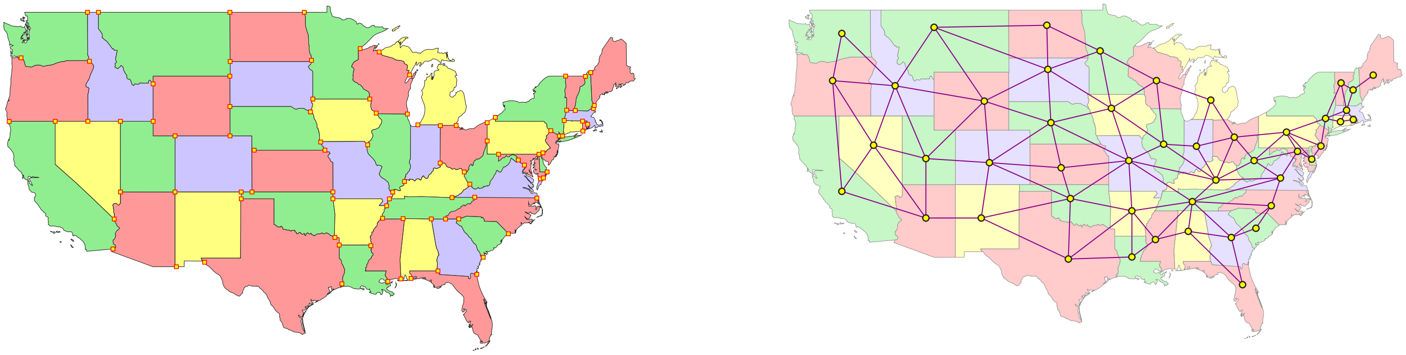

Figure 2.2: Left: A four coloring of the map of the contiguous United States. The edges are the intersections of boundaries of states, and the vertices are the endpoints of the edges. The states are thus the faces of this graph. On the right, the associated dual graph (without the outer vertex corresponding to the outer face of the original graph).

2.1.4. Duality of planar graphs

Given a drawing of a planar graph G, it naturally defines a ***dual graph***, denoted by G^*. The faces are the vertices, and two vertices (i.e., original faces) have an edge if the original faces share a boundary edge. Since the dual graph has a natural drawing as a planar graph – the dual graph is also dual. It is not hard to verify that the dual of the dual graph is the original graph. This is illustrated in Figure 2.2.