Chapter 2

Convexity

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DREAM INTERPRETATION

Simplified. Everything's either concave or convex, so whatever you dream will be something with sex.
– Piet Hein.

Version: 0.1

2.1. Some properties of convex sets

We here introduce three theorems about convex sets and convexity, which turns out to be quite useful.

Theorem 2.1.1 (Radon's Theorem). *Let $P \subseteq \mathbb{R}^d$ be a set of $n \geq d+2$ points in $\mathbb{R}^d$. Then one can partition P into to two disjoint sets X, Y, such that $P = X \cup Y$, $X \cup Y = \emptyset$, and $\mathcal{CH}(X) \cap \mathcal{CH}(Y) \neq \emptyset$.*

Proof: Proof by drawing of the 2d case. Higher dimension is proved in Section 2.2. ■

Theorem 2.1.2 (Carathéodory's Theorem). *Let P be a set of n points in $\mathbb{R}^d$, and let $p \in \mathcal{CH}(P)$ be an arbitrary point. Then P can be written as the convex combination of $d+1$ points of P.*

Proof: We prove the 2d case. Let $X = \mathcal{CH}(P)$. Since X is a convex polygon, let v its bottom vertex. Connect v to all the other vertices of X. This partitions X into triangles, and one of them contains p. As such, the point p is in the convex hull of the three vertices of this triangle, which means that p can be written as a convex combination of only these three points.

The result for higher dimensions follows from Radon's Theorem, and is omitted here. ■

2.1.1. Helly's theorem

Lemma 2.1.3. *Let $\mathcal{F}$ be a set of four convex sets S_1, S_2, S_3, S_4 in the plane, such that any three of them have a non-empty intersection. Then, all the convex sets have a non-empty intersections.*

Proof: Let p_{-i} (or simply $-i$) denote any point that lies in $\cap_{k \in \{1,2,3,4\} - i} S_k$. The four points $p_{-1}, p_{-2}, p_{-3}, p_{-4}$, by Radon's theorem, can be decomposed into two sets X and Y, such that $\mathcal{CH}(X) \cap \mathcal{CH}(Y) \neq \emptyset$.

There are two possibilities:

Figure 2.1

(A) $|X| = |Y| = 2$. For concreteness, let $X = p_{-1}, p_{-3}$ and $X = p_{-2}, p_{-4}$. Observe that $p_{-1} \in S_2$ and $p_{-3} \in S_2$. So the segment $p_{-1}p_{-3} = \mathcal{CH}(X) \subseteq S_2$, by the convexity of S_2. Similarly, $\mathcal{CH}(X) \subseteq S_4$. Namely, $\mathcal{CH}(X) \subseteq S_2 \cap S_4$. The same argument implies that $\mathcal{CH}(Y) \subseteq S_1 \cap S_3$.

We conclude that $\mathcal{CH}(X) \cap \mathcal{CH}(Y)$ (which is not empty) is contained in $S_1 \cap S_2 \cap S_3 \cap S_4$, as claimed.

(B) $X = \{-1, -2, -4\}$ and $Y = \{-3\}$. But then S_3 must contain the triangle $\triangle p_{-1}p_{-2}p_{-4}$, which in turn implies that $p_{-3} \in \triangle p_{-1}p_{-2}p_{-4} \subseteq S_3$. Namely, p_{-3} is contained in all the four convex objects. ■

With an easy trick (which one?), this leads to the following.

Lemma 2.1.4 (Helly's theorem in the plane). *Let $\mathcal{F}$ be a set of $n \geq 4$ convex sets in the plane, such that any three of them have a non-empty intersection. Then, all the convex sets in $\mathcal{F}$ have a non-empty intersections.*

Proof: The proof is by induction on n. The above lemma proves the claim for $n = 4$. So, let $\mathcal{F} = \{S_1, \ldots, S_n\}$, with $n \geq 5$. Let $T_i = S_i \cap \bigcup_{k=5}^{n} S_k$, for $i = 1, \ldots, 4$. The intersection of convex regions is convex, which implies that the T_is are convex.

Observe that the intersection of any three of the T_is corresponds to an intersection of $n-1$ of the original objects of $\mathcal{F}$. By induction on n, any such intersection of $n-1$ objects it not empty. Lemma 2.1.3 now implies that $\cap_{i=1}^{4} T_i = \cap_{i=1}^{n} S_i \neq \emptyset$. ■

The above proof can be generalized in a straightforward fashion to higher dimensions (using Radon's higher dimension variant), yielding the following.

Theorem 2.1.5 (Helly's Theorem). *Let $\mathcal{F}$ be a set of $n \geq d+2$ convex sets in $\mathbb{R}^d$, such that any $d+1$ of them have a non-empty intersection. Then, all the convex sets in $\mathcal{F}$ have a non-empty intersections.*

2.2. Proof of Radon's Theorem in higher dimensions

Claim 2.2.1. *Let $P = \{p_1, \ldots, p_{d+2}\}$ be a set of $d+2$ points in $\mathbb{R}^d$. There are real numbers $\beta_1, \ldots, \beta_{d+2}$, not all of them zero, such that $\sum_i \beta_i p_i = 0$ and $\sum_i \beta_i = 0$.*

Proof: Indeed, set $q_i = (p_i, 1)$, for $i = 1, \ldots, d+2$. Now, the points $q_1, \ldots, q_{d+2} \in \mathbb{R}^{d+1}$ are linearly dependent, and there are coefficients $\beta_1, \ldots, \beta_{d+2}$, not all of them zero, such that $\sum_{i=1}^{d+2} \beta_i q_i = 0$. Considering only the first d coordinates of these points implies that $\sum_{i=1}^{d+2} \beta_i p_i = 0$. Similarly, by considering only the $(d+1)$st coordinate of these points, we have that $\sum_{i=1}^{d+2} \beta_i = 0$. ■

Theorem 2.2.2 (Radon's theorem). *Let $P = \{p_1, \ldots, p_{d+2}\}$ be a set of $d+2$ points in $\mathbb{R}^d$. Then, there exist two disjoint subsets C and D of P, such that $\mathcal{CH}(C) \cap \mathcal{CH}(D) \neq \emptyset$ and $C \cup D = P$.*

Proof: By Claim 2.2.1 there are real numbers $\beta_1, \ldots, \beta_{d+2}$, not all of them zero, such that $\sum_i \beta_i p_i = 0$ and $\sum_i \beta_i = 0$.

Assume, for the sake of simplicity of exposition, that $\beta_1, \ldots, \beta_k \geq 0$ and $\beta_{k+1}, \ldots, \beta_{d+2} < 0$. Furthermore, let $\mu = \sum_{i=1}^{k} \beta_i = -\sum_{i=k+1}^{d+2} \beta_i$. We have that

$$\sum_{i=1}^{k} \beta_i p_i = -\sum_{i=k+1}^{d+2} \beta_i p_i.$$

In particular, $v = \sum_{i=1}^{k} (\beta_i/\mu) p_i$ is a point in $\mathcal{CH}(\{p_1, \ldots, p_k\})$. Furthermore, for the same point v we have $v = \sum_{i=k+1}^{d+2} -(\beta_i/\mu) p_i \in \mathcal{CH}(\{p_{k+1}, \ldots, p_{d+2}\})$. We conclude that v is in the intersection of the two convex hulls, as required. ■

2.3. Bibliographical notes

The material here is pretty standard – see Chapter 1 in Matoušek [Mat02].

References

[Mat02] J. Matoušek. *Lectures on Discrete Geometry*. Vol. 212. Grad. Text in Math. Springer, 2002.