Generalized Quantum States

via Mixed States and Density Operators

Fernando Granha Jeronimo

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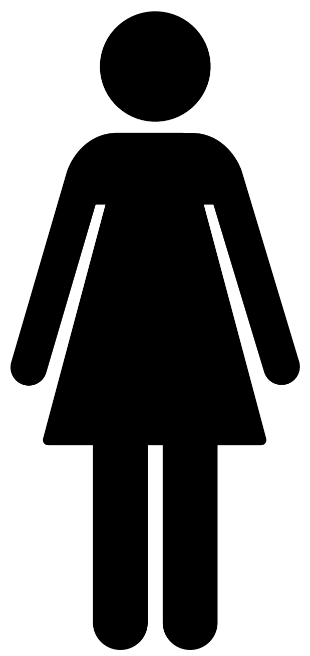
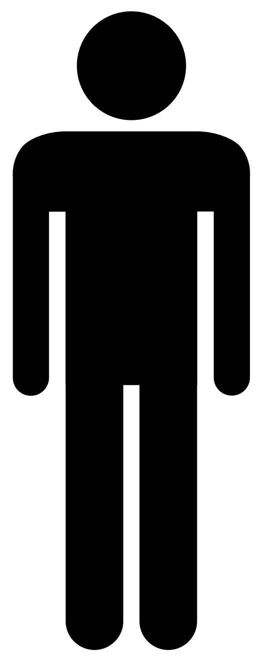
Is this kind of description sufficient?

Scenario 1

Suppose Alice and Bob share an EPR pair





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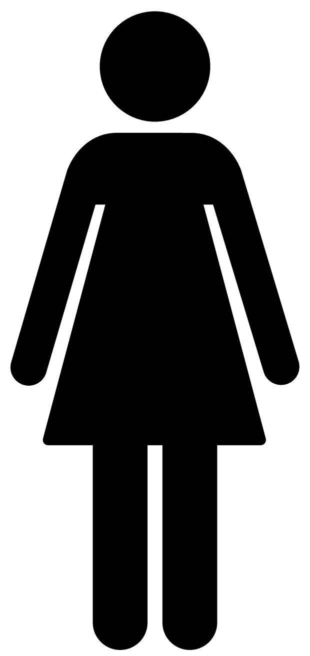

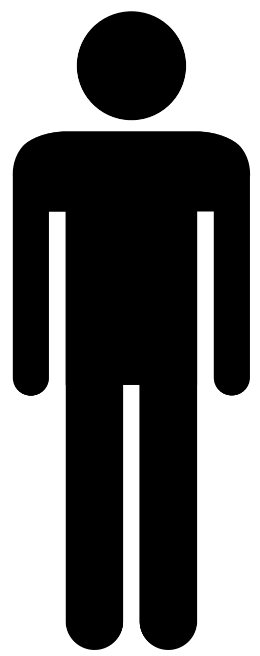

$$|\textbf{EPR}\rangle = \frac{1}{\sqrt{2}}|0\rangle^A|0\rangle^B + \frac{1}{\sqrt{2}}|1\rangle^A|1\rangle^B$$

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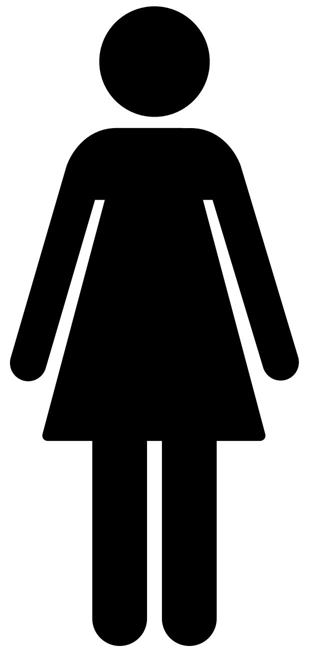

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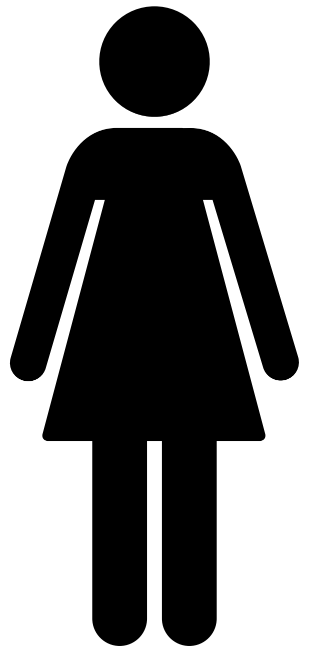

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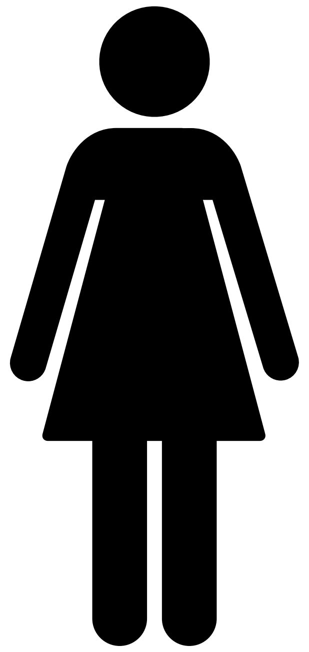

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It is not a pure state!

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Scenario 2

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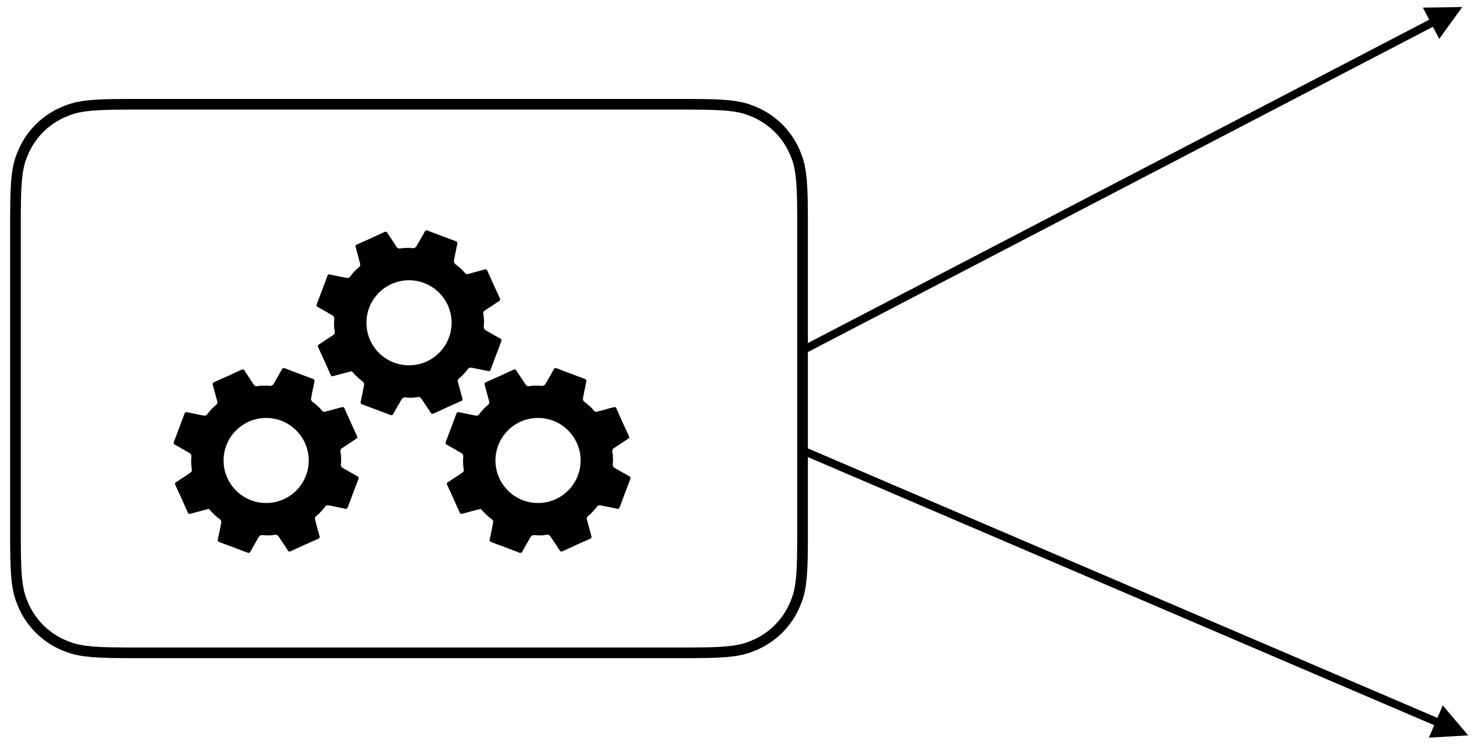

$|0\rangle$ with probability $\frac{1}{2}$

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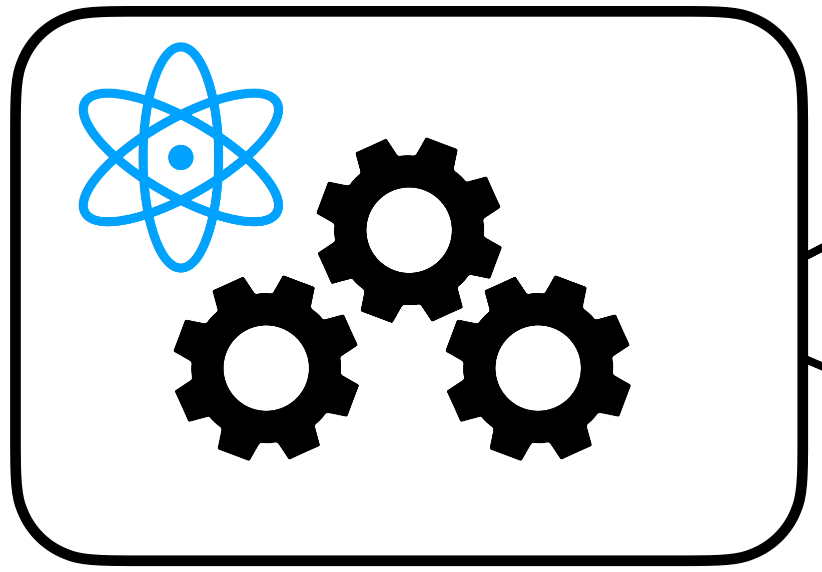
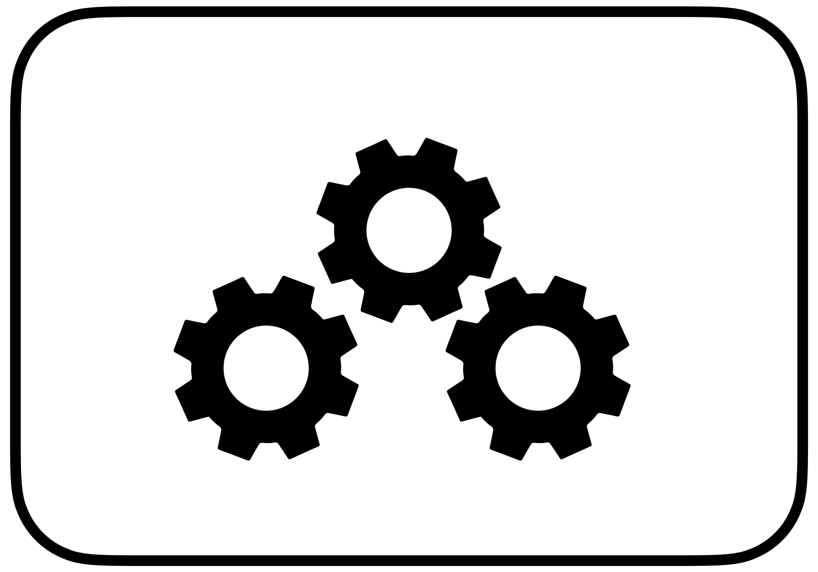


$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad \textbf{with probability } \frac{1}{2}$$

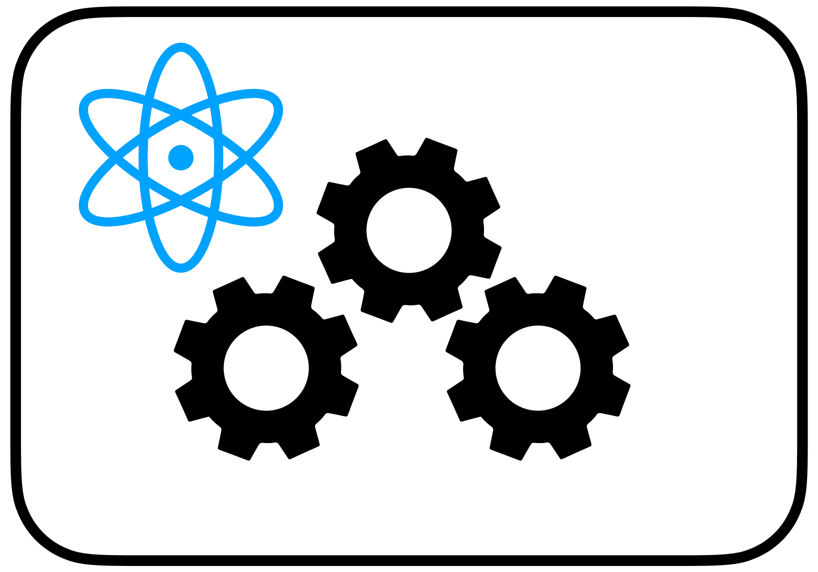
$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \quad \textbf{with probability } \frac{1}{2}$$

How can we describe the output quantum state?

How different are the outputs of these machines?



Classical



Quantum

We need a more general formalism to describe quantum states!

The Ensemble Formalism

$[m] = \{1, 2, \ldots, m\}$

Given

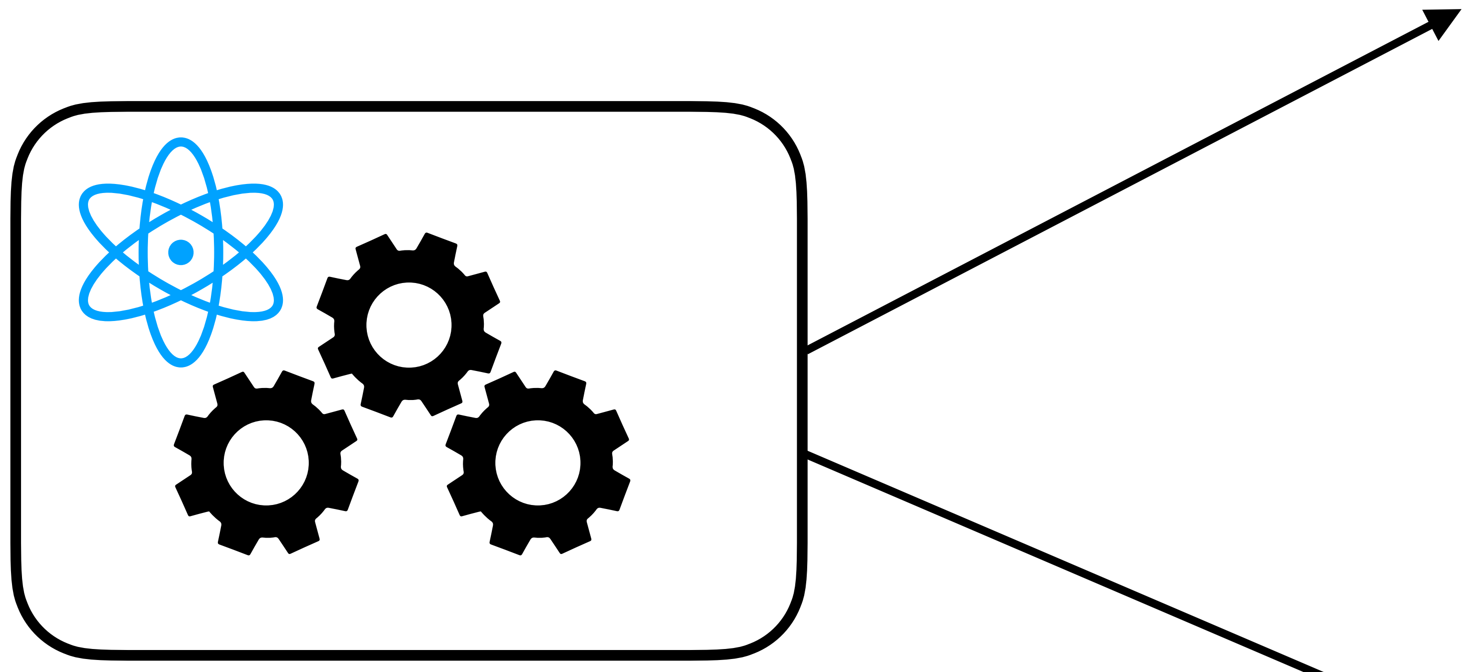
(1) $\{p_i\}_{i\in[m]}$ **probability distribution** $\left(\sum_{i=1}^{m} p_i = 1, p_i \geq 0 \forall i \in [m]\right)$ **and**

(2) $\{|\psi_i\rangle \in \mathbb{C}^d\}_{i\in[m]}$ **pure quantum states,**

we define the *ensembe* of quantum states:

$$\left\{ \left(p_i, |\psi_i\rangle\right) \right\}_{i\in[m]}$$

Exemple of Ensembles



$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ **with probability** $\frac{1}{2}$

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$$\left\{\left(\frac{1}{2}, |+\rangle\right), \left(\frac{1}{2}, |-\rangle\right)\right\}$$

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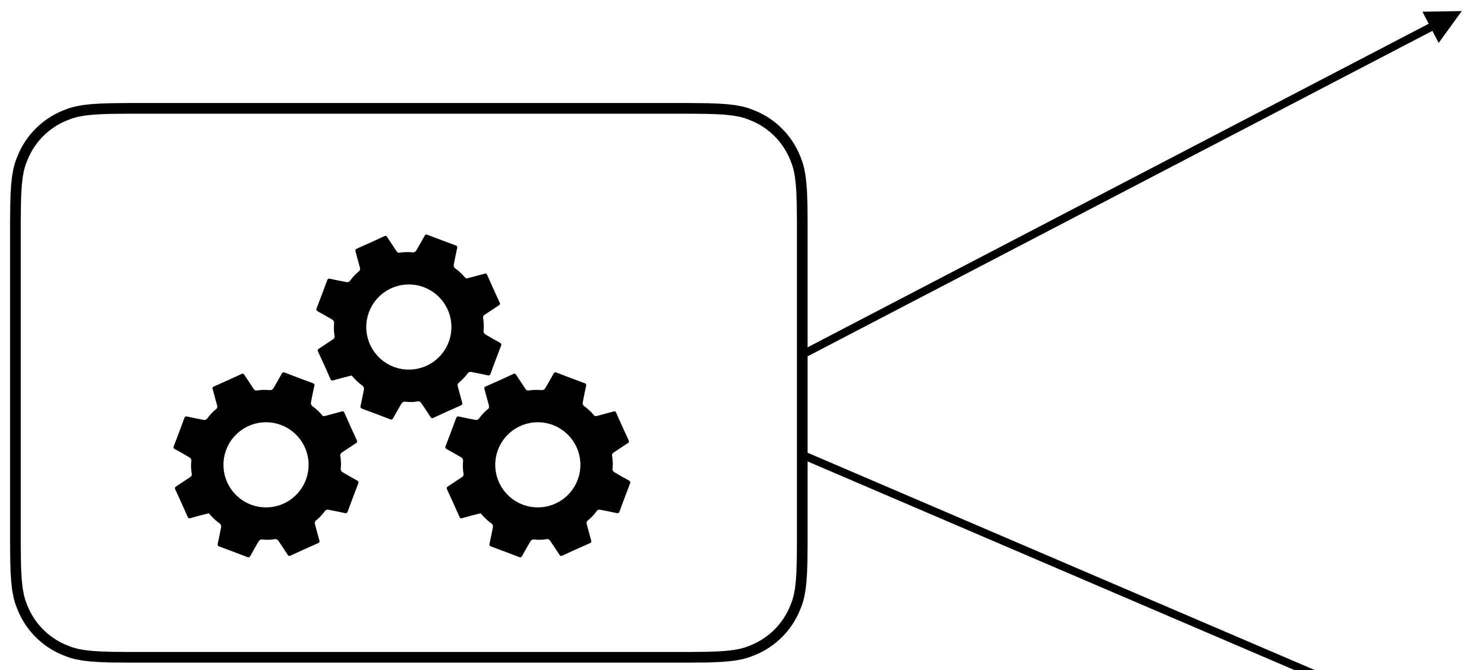

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Exemple of Ensembles

$|\psi\rangle \in \mathbb{C}^d$ **(a pure quantum state)**

$$\left\{\left(1, |\psi\rangle\right)\right\}$$

Density Operator Formalism

A quantum state is a matrix $\rho \in \mathbb{C}^{d\times d}$ satisfying

(1) **Tr**$(\rho) = 1$

(2) $\rho \succeq 0$ **positive semi-definite (PSD), i.e.,** $\rho = \rho^\dagger$ **and** $\langle\phi|\rho|\phi\rangle \geq 0, \forall |\phi\rangle \in \mathbb{C}^d$

From Ensembles to Density Operators

Given an ensemble $\left\{ \left(p_i, |\psi_i\rangle\right) \right\}_{i \in [m]}$ **, what is the corresponding density operator?**

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(3) $$\langle\phi|\rho|\phi\rangle = \sum_{i=1}^{m} p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle = \sum_{i=1}^{m} p_i |\langle\phi|\psi_i\rangle|^2 \geq 0, \forall |\phi\rangle \in \mathbb{C}^d$$ ✓

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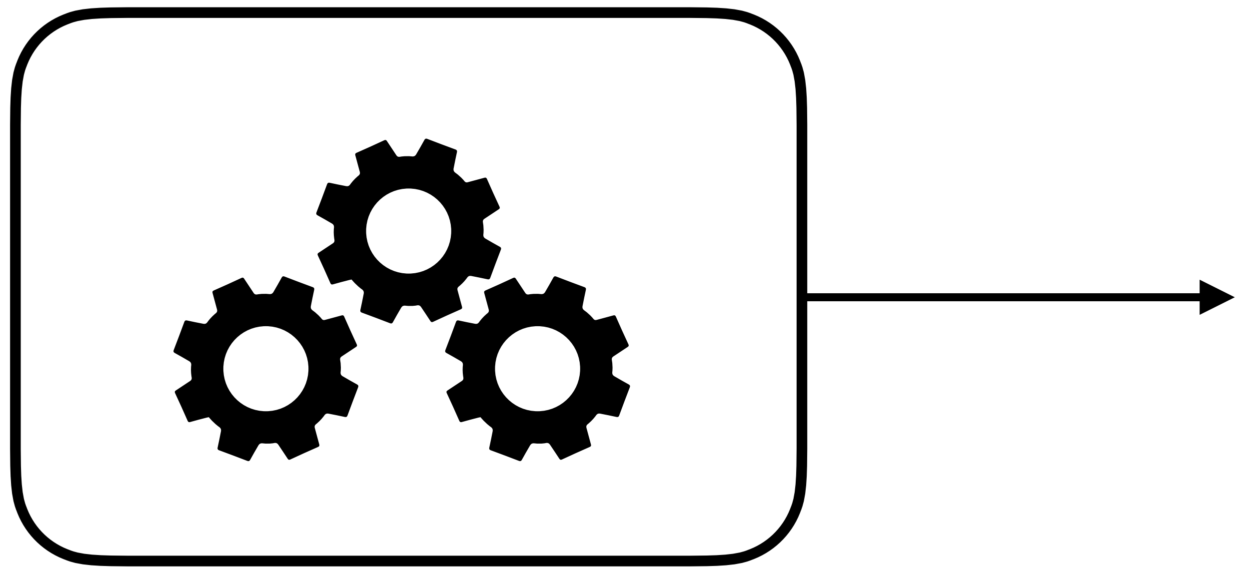

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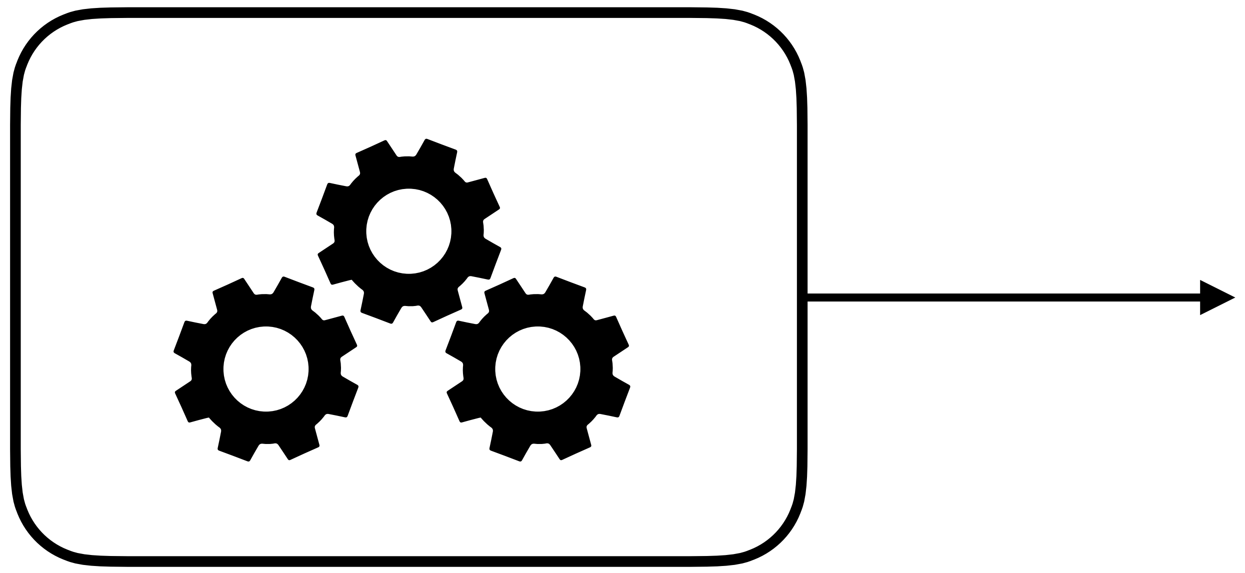

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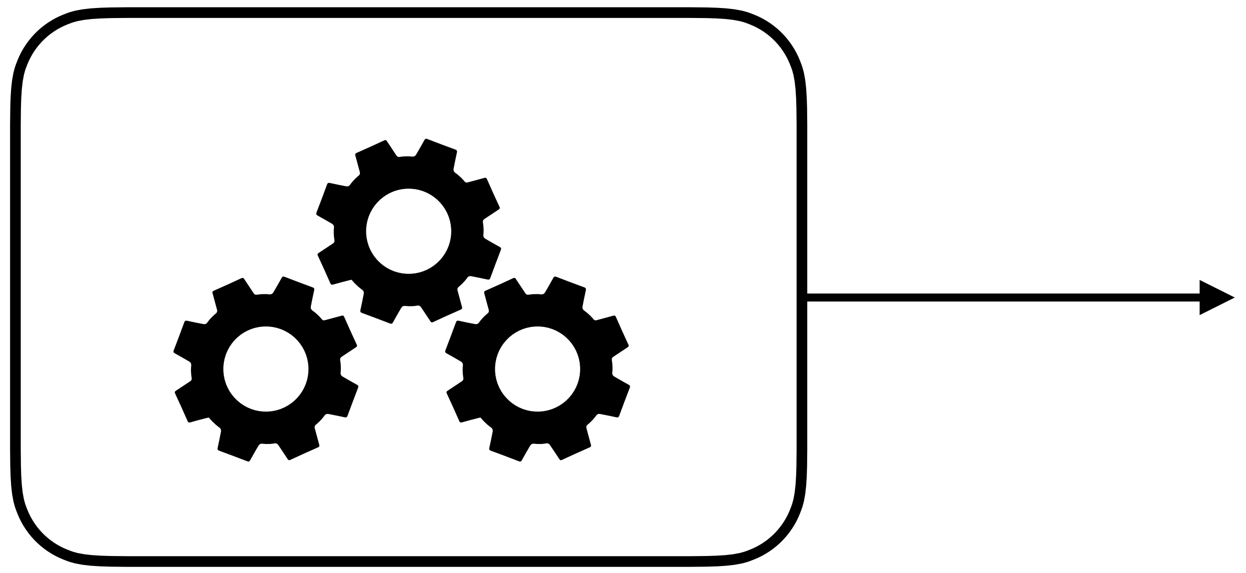

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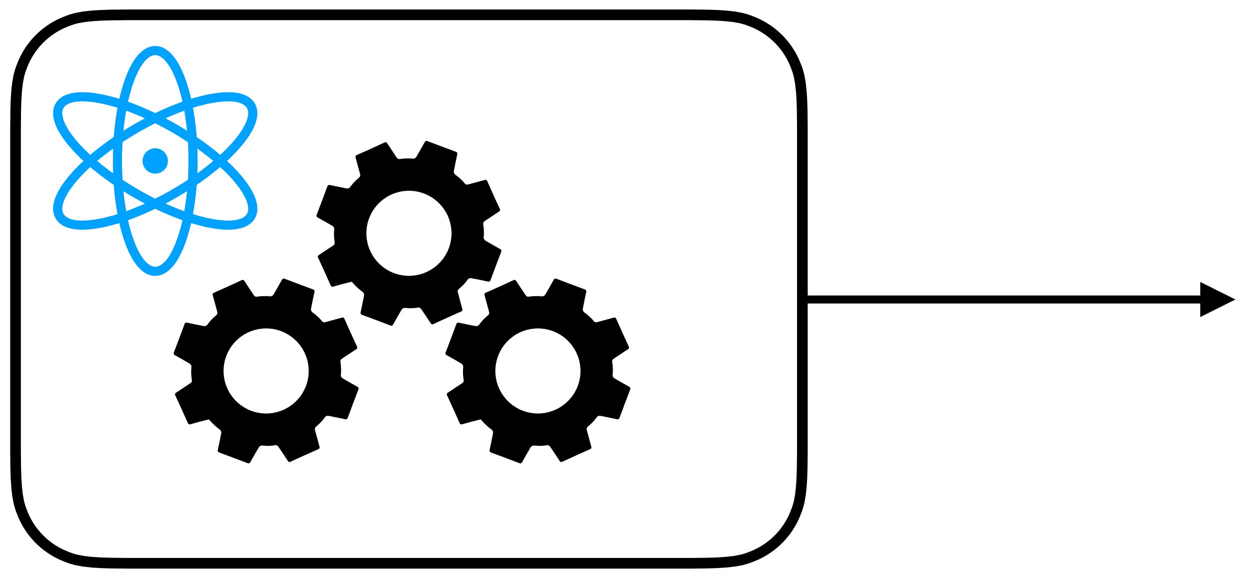

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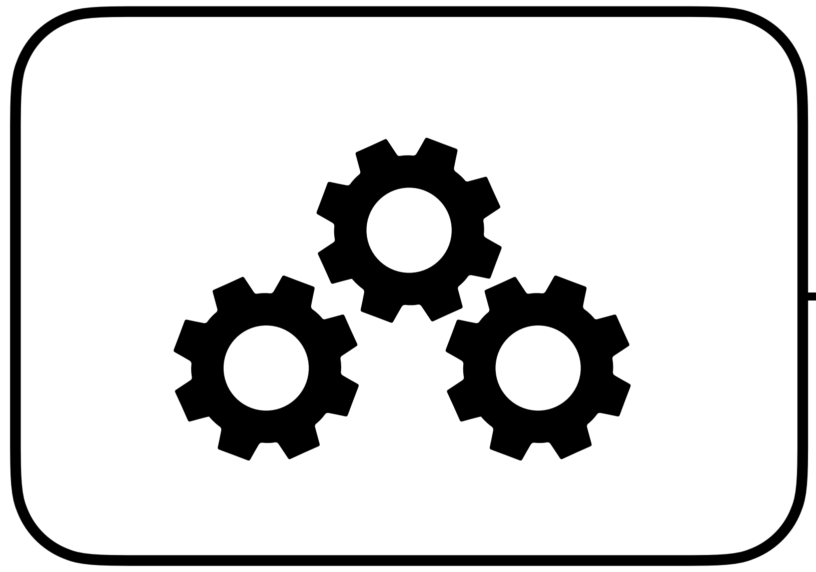

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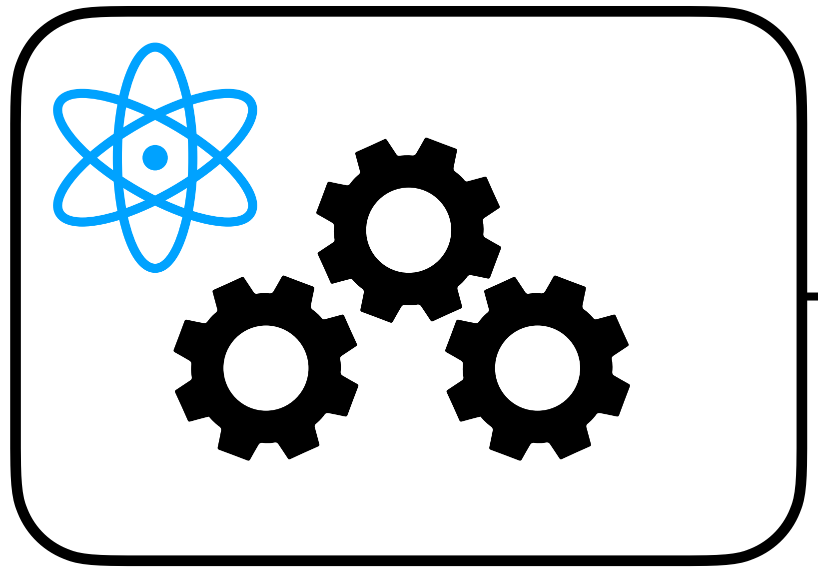

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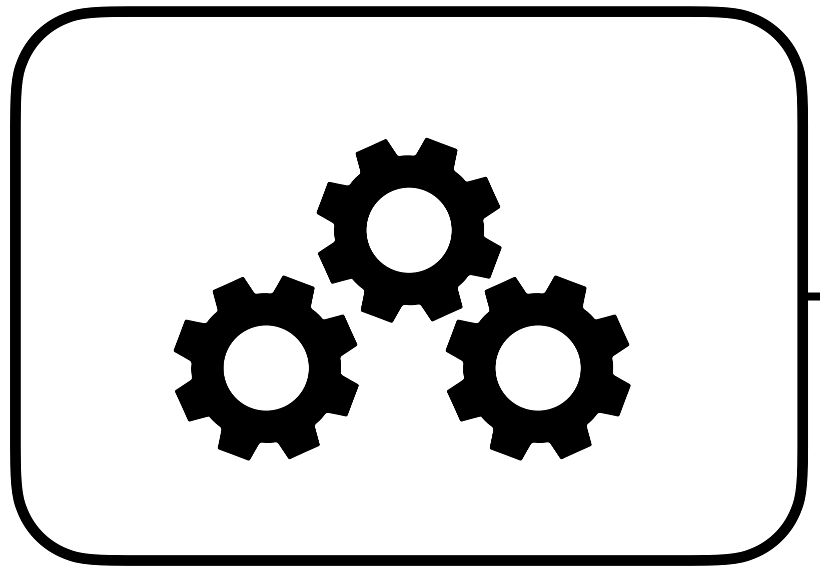

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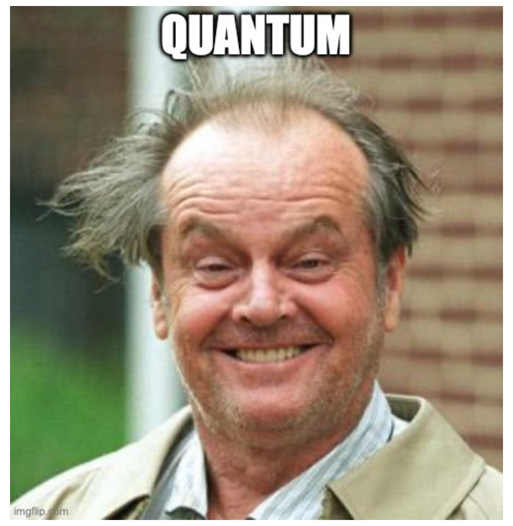

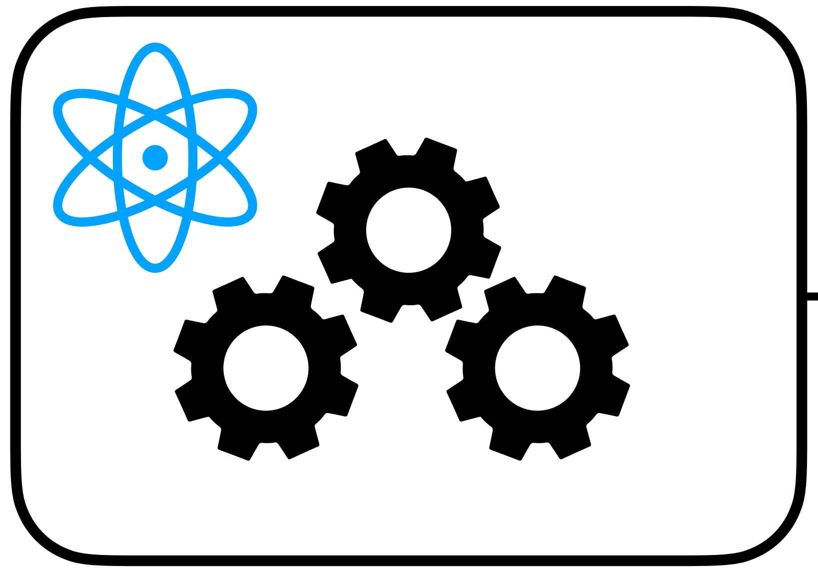

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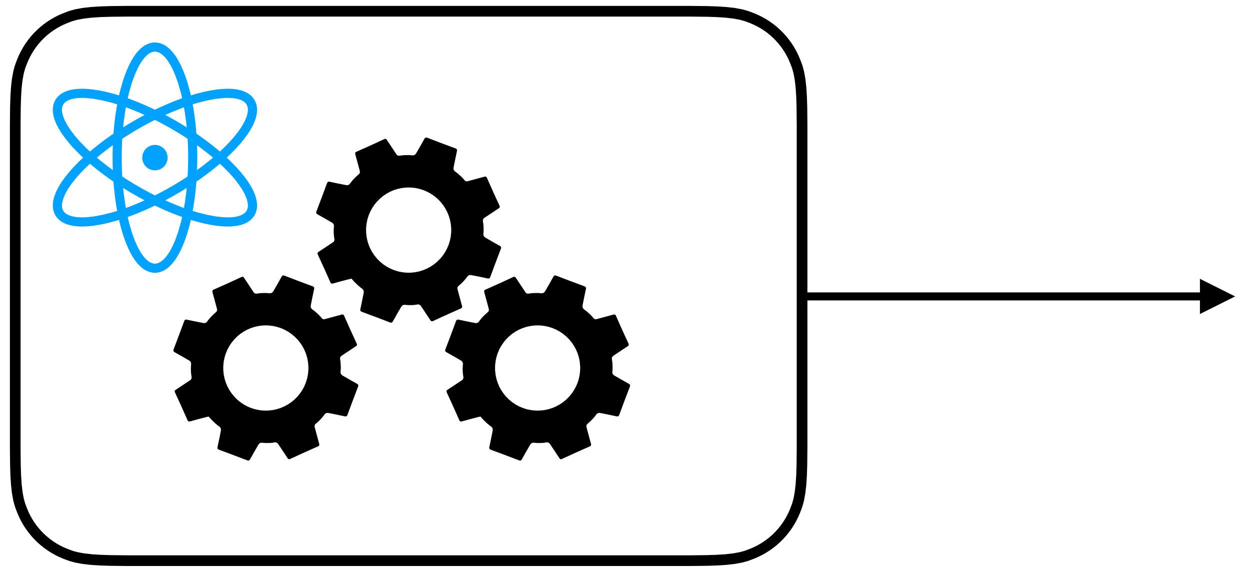

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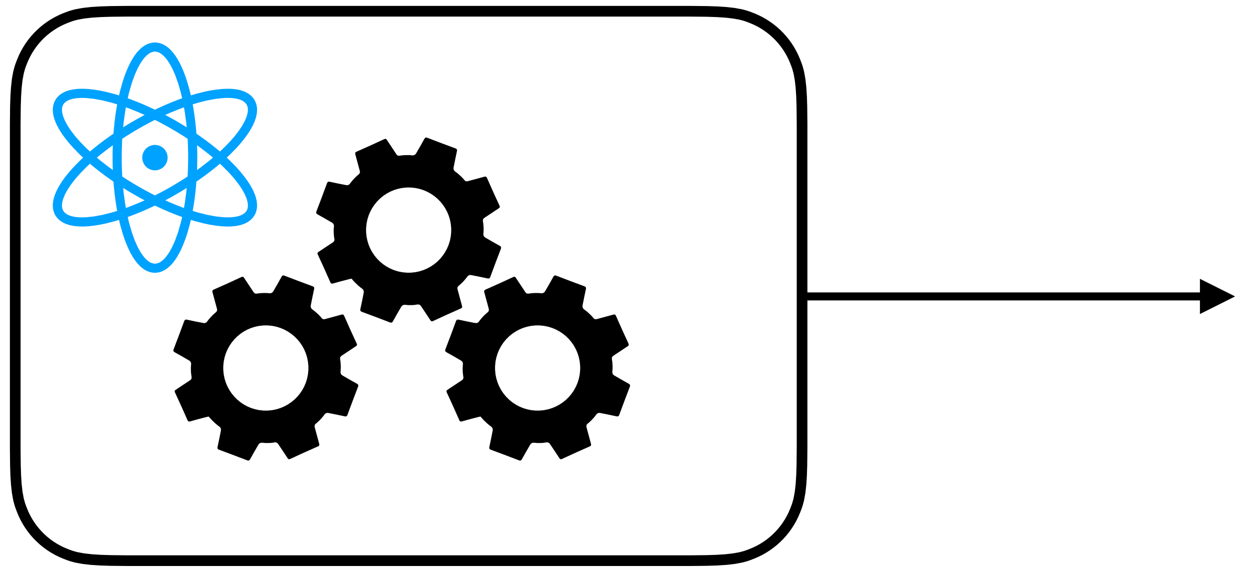

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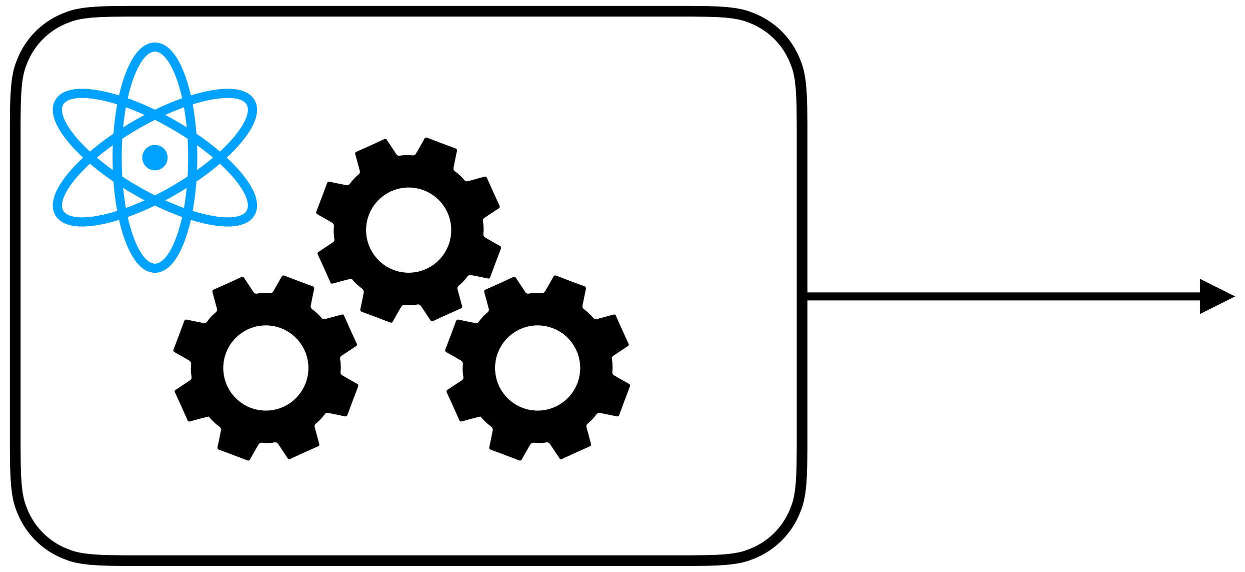

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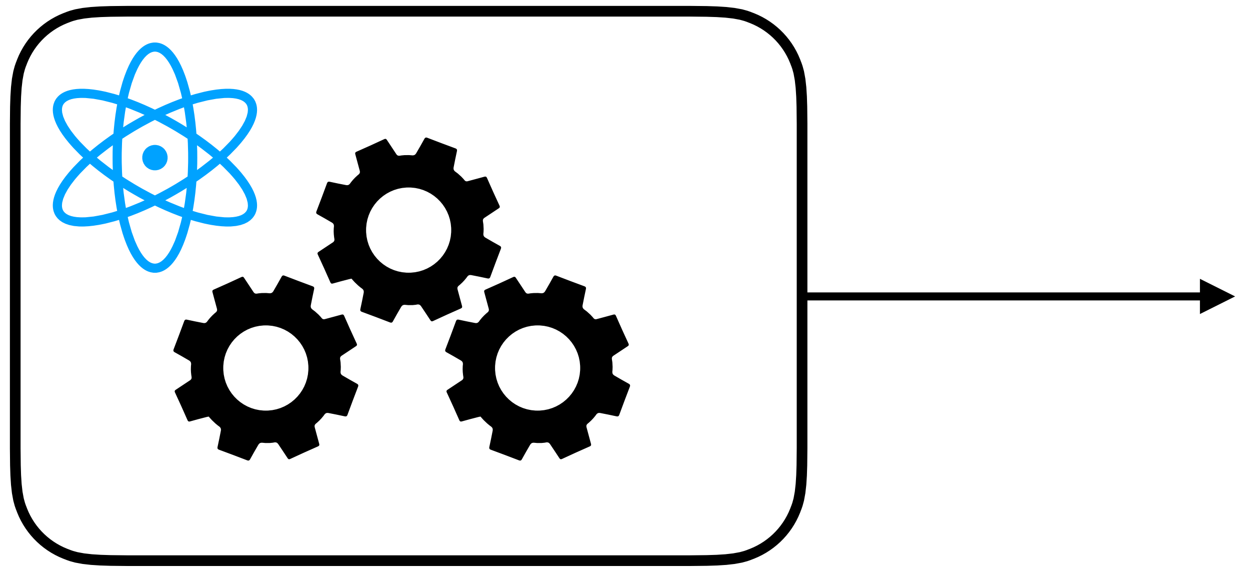

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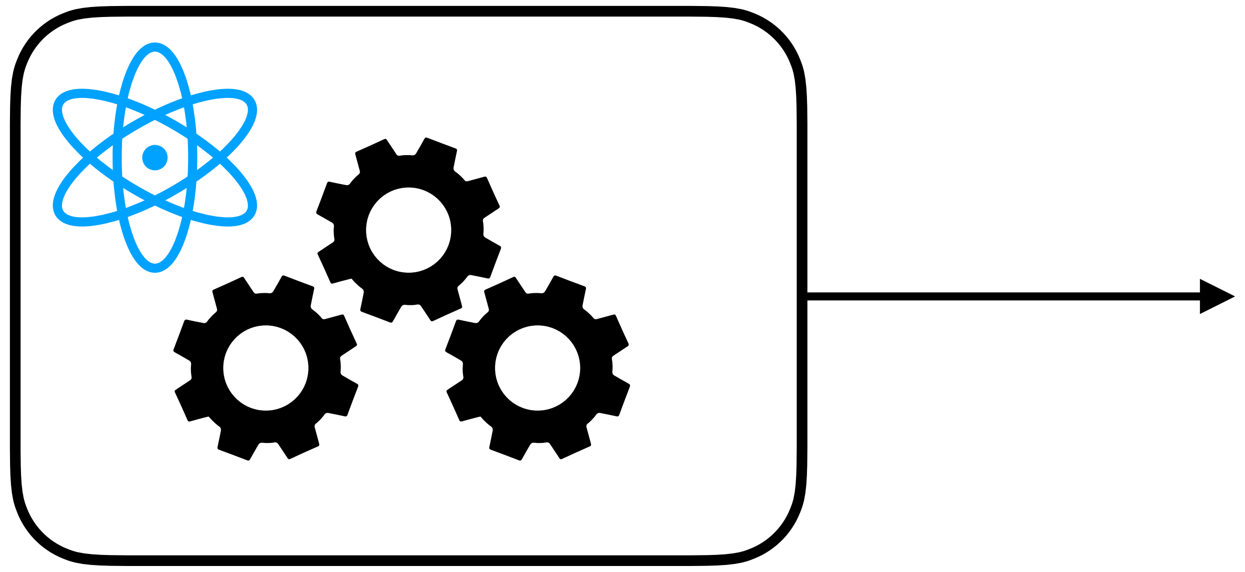

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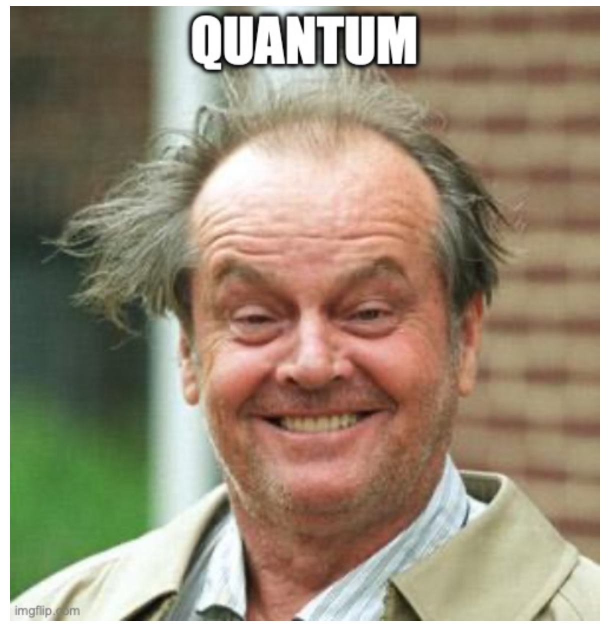

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For this, we will invoke the Spectral Theorem from Linear Algebra!

Refresher on the Spectral Theorem

[Spectral Theorem]

Given a normal matrix $A \in \mathbb{C}^{d \times d}$ **(i.e.,** $AA^\dagger = A^\dagger A$**), there exist**

(1) **Orthonomal basis of eigenvectors** $\{|\psi_i\rangle\}_{i \in [d]}$**, and**

(2) **Correspoding Eigenvalues** $\{\lambda_i\}_{i \in [d]}$ **(i.e.,** $A|\psi_i\rangle = \lambda_i|\psi_i\rangle$**), such that**

$$A = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|$$

Refresher on the Spectral Theorem

In particular, this theorem holds for Hermitian and unitary matrices

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$$\Big\{ \big(\lambda_i, |\psi_i\rangle\big) \Big\}_{i\in[d]}$$ **ensemble**

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$$\rho = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|$$

(1) $\mathbf{Tr}(\rho) = \sum_{i=1}^{d} \lambda_i = 1$

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(2) $\rho = \rho^\dagger \implies \lambda_i \in \mathbb{R}, \forall i \in [d]$

(3) $\rho \geq 0 \implies \lambda_i \geq 0, \forall i \in [d]$

$\downarrow$

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ensemble

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Unitary Evolution

Given unitary U, evolution is given by

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(Ensemble)

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(Ensemble)

$$\rho \mapsto_U U\rho U^\dagger$$

(Density Operator)

Measurement

A measurement is defined by a collection $\{M_i = E_i^\dagger E_i \in \mathbb{C}^{d\times d}\}_{i\in[m]}$ **of positive semi-definite (PSD) matrices sastisfying** $\sum_{i=1}^{m} M_i = I$

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$$(2) \quad \sum_{i=1}^{m} M_i = I, \mathrm{Tr}(\rho) = 1 \implies \sum_{i=1}^{m} p_i = \mathrm{Tr}\left(\sum_{i=1}^{m} M_i\rho\right) = 1 \quad ✓$$

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(we require an explicit factorization $M_i = E_i^\dagger E_i$ since $M_i = E_i^\dagger U^\dagger U E_i$ for any unitary U)

POVM Measurement

The collection $\{M_i \in \mathbb{C}^{d\times d}\}_{i\in[m]}$ of positive semi-definite (PSD) matrices sastisfying $\sum_{i=1}^{m} M_i = I$

is called <u>P</u>ositive <u>O</u>perator-<u>V</u>alued <u>M</u>easure (POVM)

(In this case we do not require an explicit factorization $M_i = E_i^\dagger E_i$)

Example of Measurement

Measuring in the computational basis $\{M_x = |x\rangle\langle x| \in \mathbb{C}^{2^n \times 2^n}\}_{x \in \{0,1\}^n}$

Suppose $\rho = |\psi\rangle\langle\psi|$, **where** $|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$

Then, $p_x =$ **Tr**$(M_x \rho) =$ **Tr**$(|x\rangle\langle x|\rho) = \langle x|\rho|x\rangle = \langle x|\psi\rangle\langle\psi|x\rangle = |\alpha_x|^2$

Example of Measurement

Measuring Hamming weight parity

$$\left\{ M_{\textbf{even}} = \sum_{x \in \{0,1\}^n:\ |x| \equiv 0 \pmod 2} |x\rangle\langle x|, M_{\textbf{odd}} = I - M_{\textbf{even}} \right\}$$

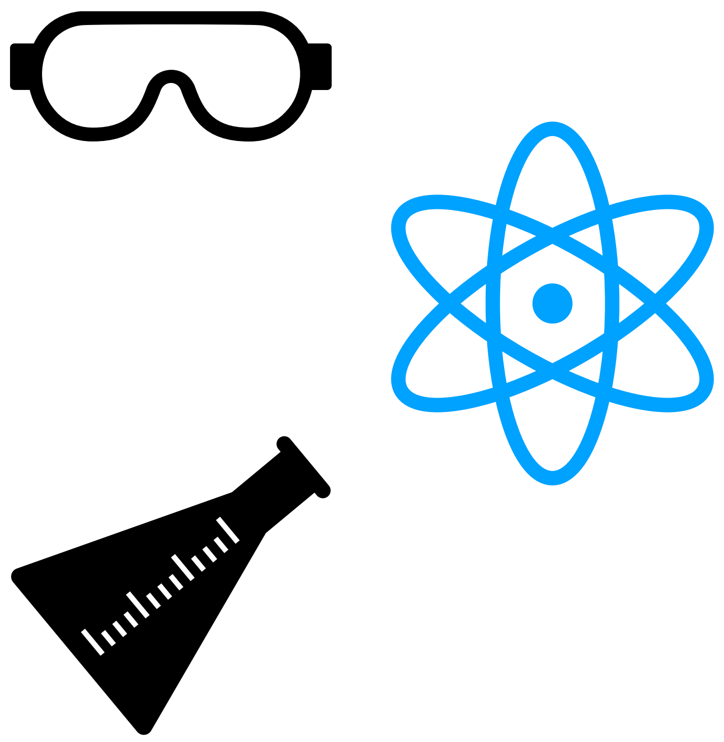
Suppose $\rho = |\psi\rangle\langle\psi|$, **where** $|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$

Then, $p_{\textbf{even}} = \textbf{Tr}(M_{\textbf{even}}\rho) = \sum_{x \in \{0,1\}^n:\ |x| \equiv 0 \pmod 2} |\alpha_x|^2$

How can we learn a quantum state?

Learning Scenario

Suppose that we have a procedure that outputs a qubit and we want to learn its state



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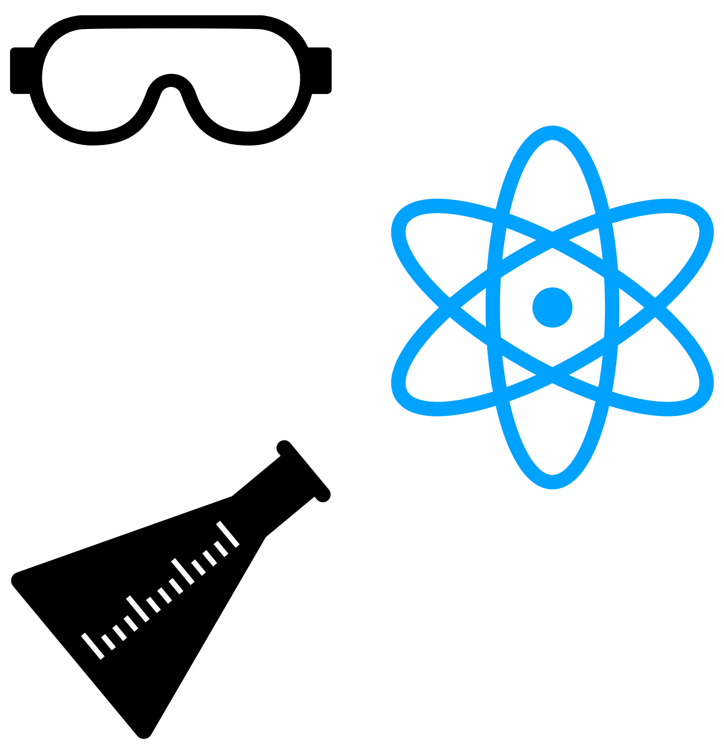

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This task is called Quantum State Tomography

Hilbert-Schmidt Inner Product

Let $A, B \in \mathbb{C}^{d \times d}$**. The Hibert-Schmidt inner product betwen** A **and** B **is defined as**

$$\langle A, B \rangle = \textbf{Tr}(A^\dagger B) = \sum_{i=1}^{d} \sum_{j=1}^{d} \overline{A_{i,j}} B_{i,j}$$

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(this is the entrywise inner product of the matrices)

Pauli Matrices

Widely used matrices in Quantum Information Sciences in many contexts

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

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$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

bit flip error

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phase flip error

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bit and phase flip erros

Pauli Matrices

Some Properties

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Pauli Matrices

Given an arbitrary Hermitian matrix $H \in \mathbb{C}^{2\times 2}$, $H = \begin{pmatrix} a & b-ic \\ b+ic & d \end{pmatrix}$**, we have**

$$H = \frac{(a+d)}{2} I + bX + cY + \frac{(a-d)}{2} Z$$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

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The matrices $\{I, X, Y, Z\}$ form an orthogonal basis for the space of Hermitian matrices in $\mathbb{C}^{2\times 2}$

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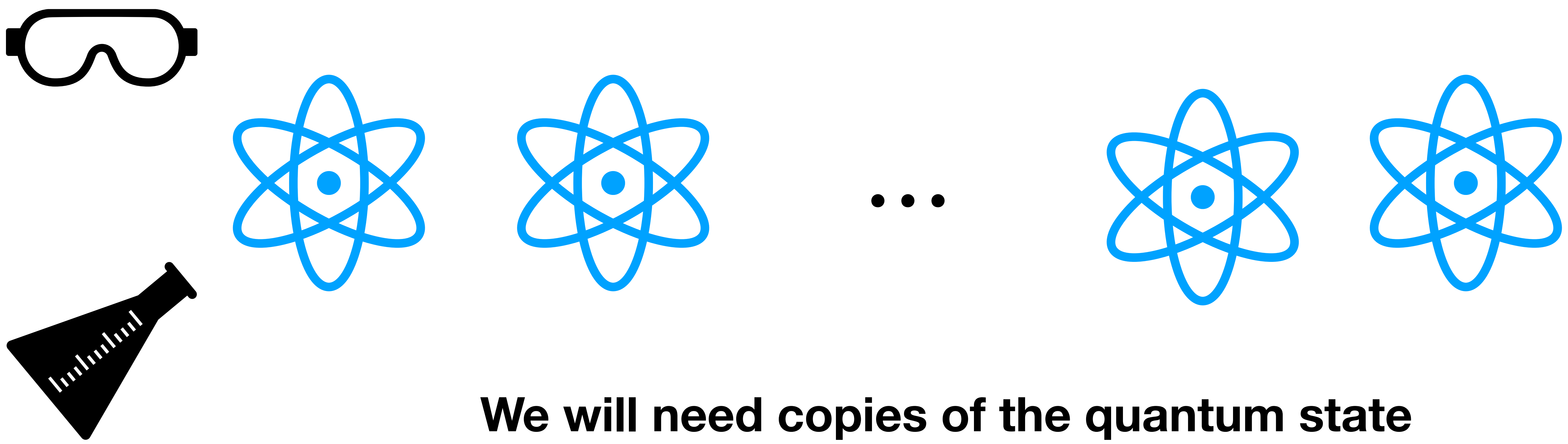

We will need copies of the quantum state

What is the quantum state $\rho \in \mathbb{C}^{2\times 2}$?

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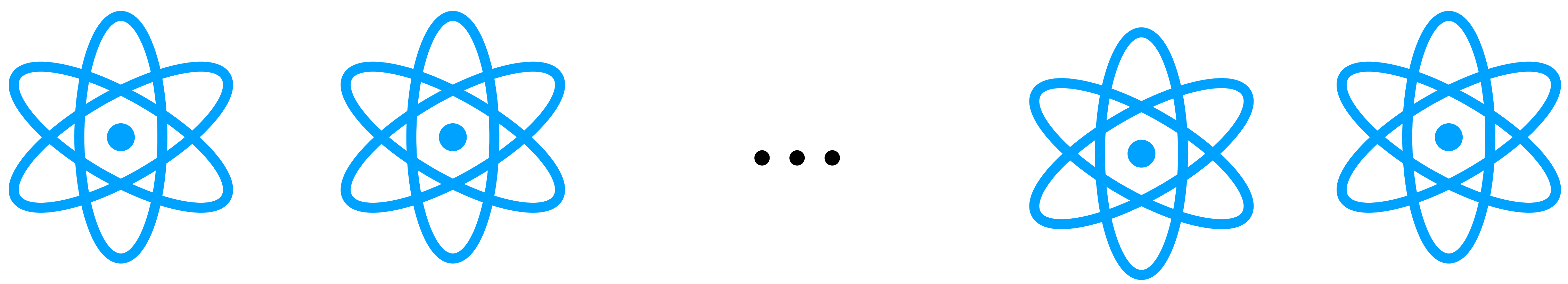

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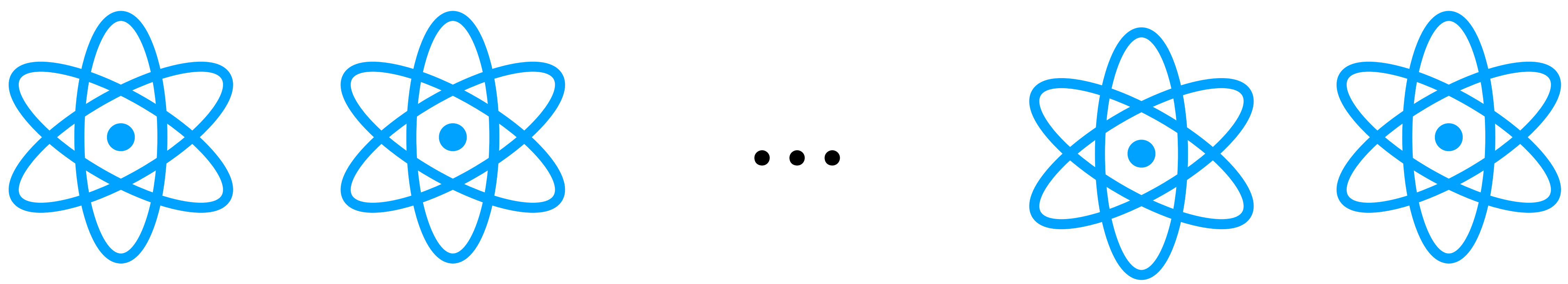

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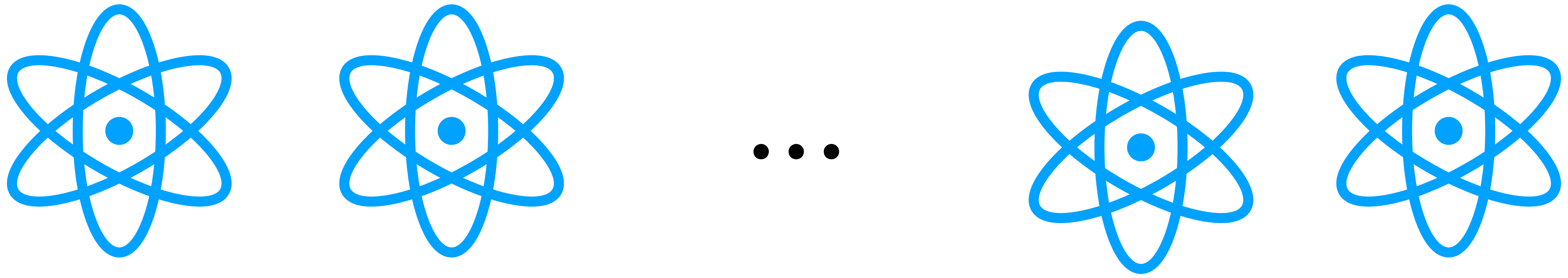

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Output the approximation

$$\rho = \frac{I + a_x X + a_y Y + a_z Z}{2} \approx \hat{\rho} = \frac{I + \hat{a}_x X + \hat{a}_y Y + \hat{a}_z Z}{2}$$

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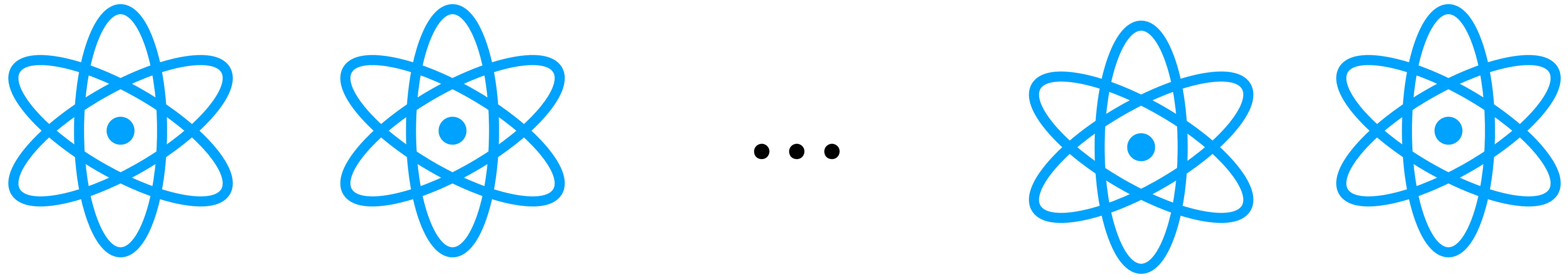
What is the quantum state $\rho \in \mathbb{C}^{2^n \times 2^n}$?



We will need maaaaa…any copies of the quantum state for n large!

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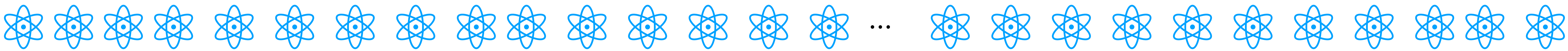

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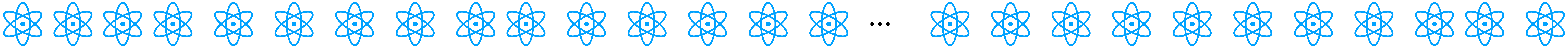

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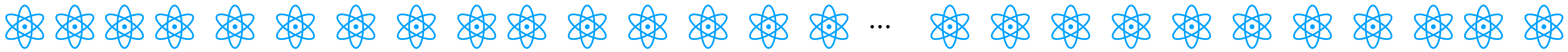

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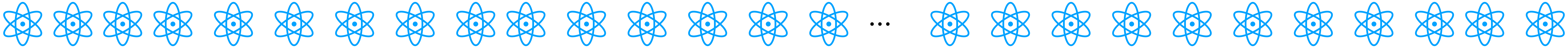

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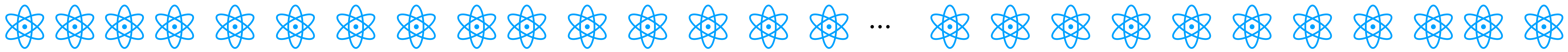

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No! Many notions of cheaper tomography known as Shadow Tomography!

A Glimplse of an Active Research Topic:
<u>The Shadow Tomography Case</u>

Shadow Tomography

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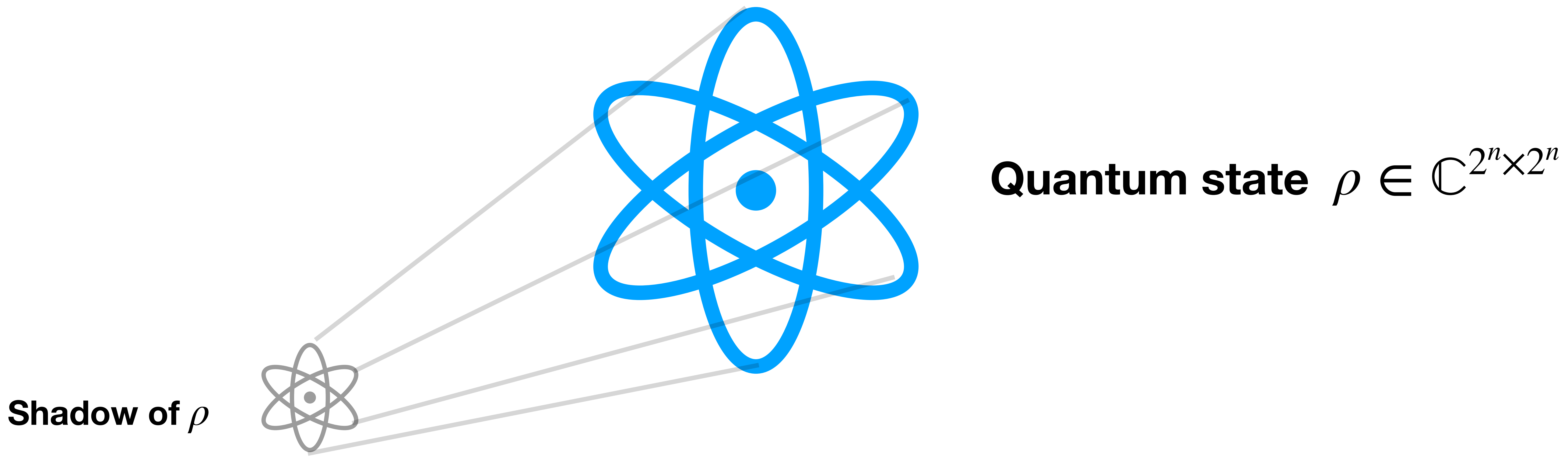

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[Aaronson'16]

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<u>Amazing</u>: In particular, we can learn $4^{\Theta(n)}$ properties (as above) of n-qubit states with only $n^{O(1)}$ copies!

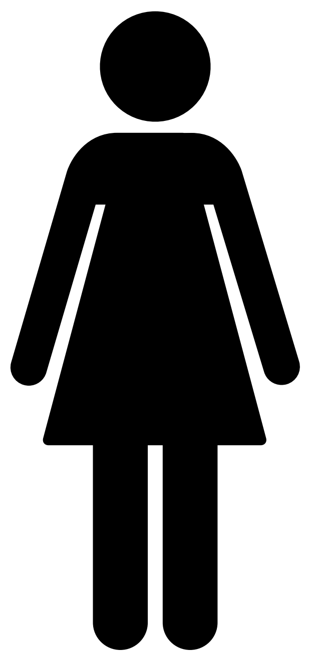
Shadow Tomography

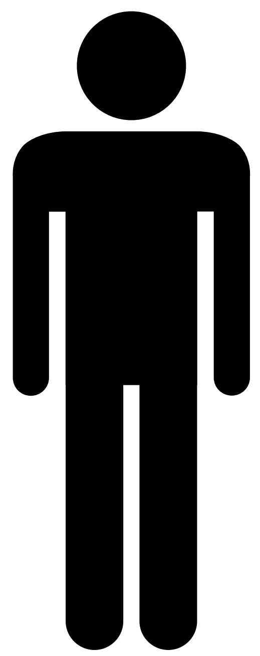
There are many variants of Shadow tomography extending the seminal work of Aaronson with various guarantees

(this is an active research area)

Back to Alice in Scenario 1

Suppose Alice and Bob share an EPR pair





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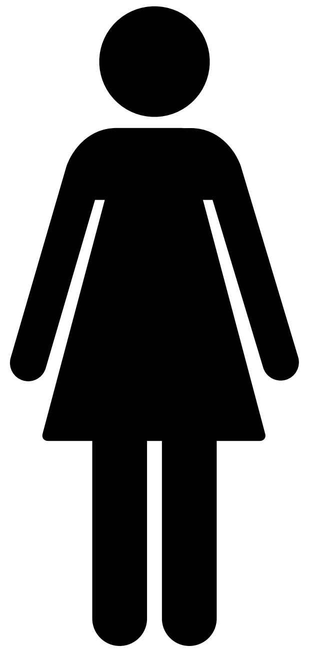

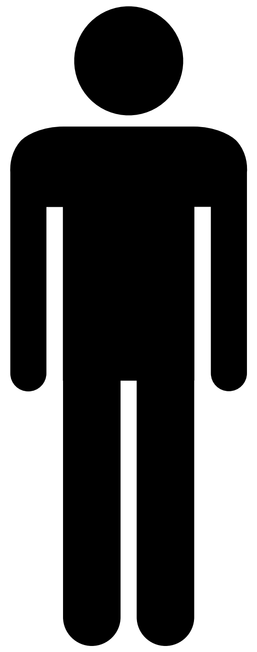

$$|\textbf{EPR}\rangle = \frac{1}{\sqrt{2}}|0\rangle^A|0\rangle^B + \frac{1}{\sqrt{2}}|1\rangle^A|1\rangle^B$$

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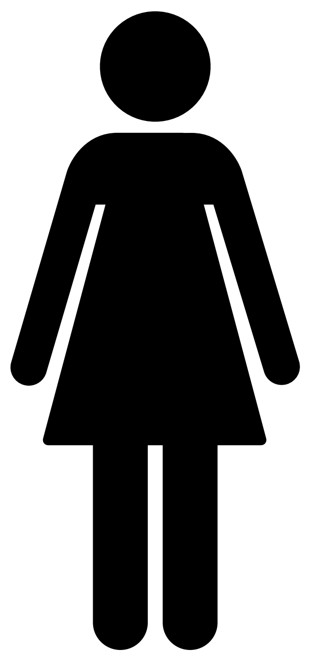

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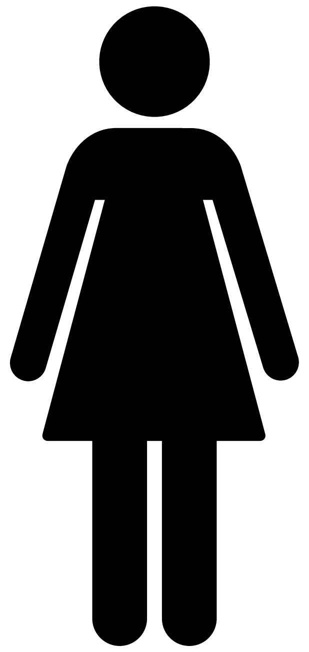

What is the state of Alice's quantum system?

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}}|0\rangle^A|0\rangle^B + \frac{1}{\sqrt{2}}|1\rangle^A|1\rangle^B \neq |\psi_A\rangle \otimes |\psi_B\rangle$$

(no matter the choice of $|\psi_A\rangle, |\psi_B\rangle$)

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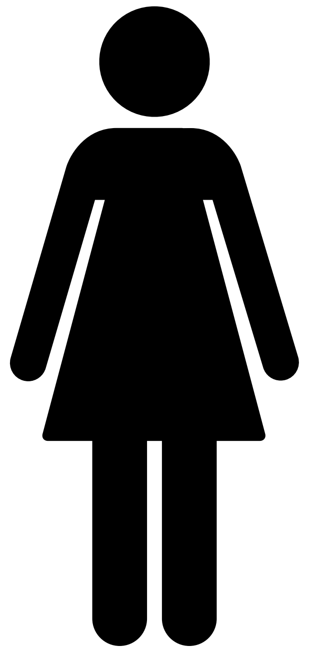

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It is not a pure state!

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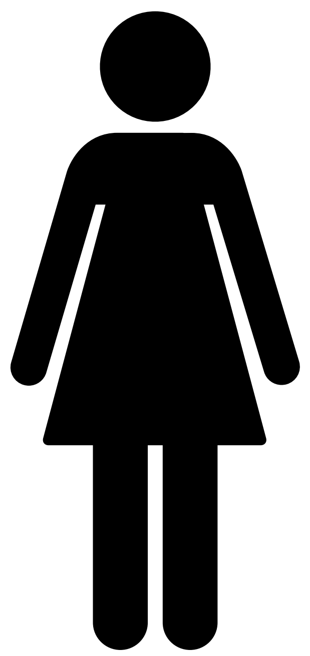
Suppose Alice and Bob share an EPR pair



What is the state of Alice's quantum system?

It is not a pure state!

It is a mixed state given by a density operator!

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2}}|0\rangle^A|0\rangle^B + \frac{1}{\sqrt{2}}|1\rangle^A|1\rangle^B \neq |\psi_A\rangle \otimes |\psi_B\rangle$$

(no matter the choice of $|\psi_A\rangle, |\psi_B\rangle$)

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We will need the partial trace operation!

Sub-system State via Partial Trace

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$\mathscr{H}_A \otimes \mathscr{H}_B$ **with orthornomal basis** $\{|i\rangle_A\}_{i\in[d_A]}$ **and** $\{|i\rangle_B\}_{i\in[d_B]}$

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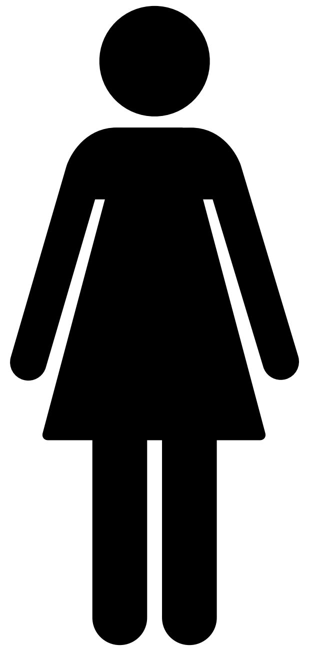
where the partial trace $\mathbf{Tr}_B(\cdot)$ **is the linear operator acting on basis elements as**

$$\mathbf{Tr}_B(|i_A\rangle\langle j_A| \otimes |i_B\rangle\langle j_B|) = |i_A\rangle\langle j_A|\mathbf{Tr}(|i_B\rangle\langle j_B|) = |i_A\rangle\langle j_A|\langle j_B|i_B\rangle$$

Alice's State via Partial Trace



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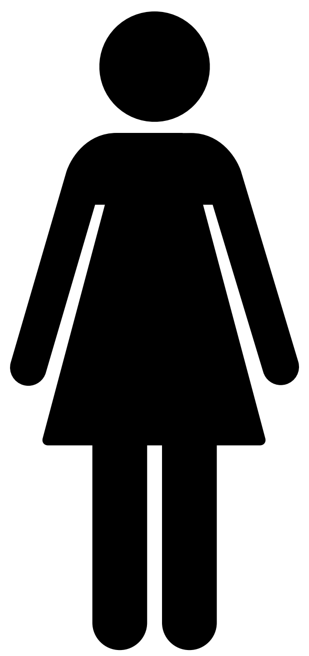

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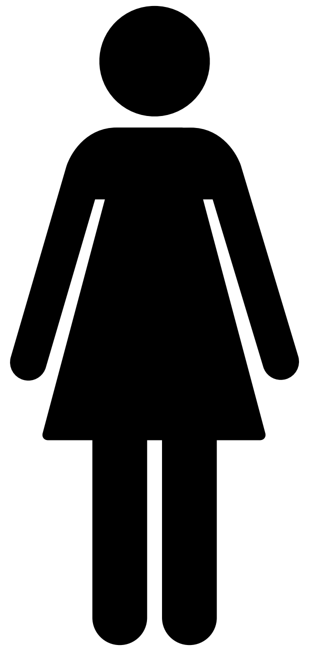
$$\begin{aligned}\rho_A &= \textbf{Tr}_B(\rho_{AB}) \\ &= \frac{1}{2}\textbf{Tr}_B(|0\rangle^A|0\rangle^B\langle 0|^A\langle 0|^B) + \frac{1}{2}\textbf{Tr}_B(|1\rangle^A|1\rangle^B\langle 0|^A\langle 0|^B) \\ &\quad + \frac{1}{2}\textbf{Tr}_B(|0\rangle^A|0\rangle^B\langle 1|^A\langle 1|^B) + \frac{1}{2}\textbf{Tr}_B(|1\rangle^A|1\rangle^B\langle 1|^A\langle 1|^B) \\ &= \frac{1}{2}|0\rangle^A\langle 0|^A + \frac{1}{2}|1\rangle^A\langle 1|^A\end{aligned}$$

Alice's State via Partial Trace



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Maximally mixed state!

Due to *entanglement* between Alice and Bob,
Alice reduced state is mixed!

What is quantum *entanglement*?

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(We will focus on bipartite entanglement of pure quantum states)

Thank you!

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More Questions?