LECTURE 7 February 11^{th}, 2025

PART I Fundamental Concepts & Applications in Quantum Information

TODAY EPR paradox, Bell's Theorem & the CHSH game

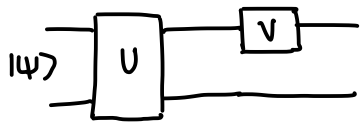
RECAP Fundamental Concepts in Quantum Information

State of multi-qubit systems $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

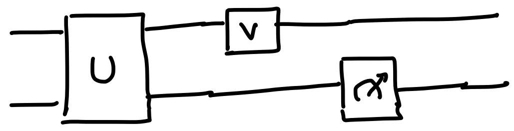
Entanglement Not of the form $|\psi\rangle \otimes |\phi\rangle$

e.g the Bell State / EPR pair $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

Unitary Transformations & Quantum Circuits



Partial Measurements & Mixed State



Mixed State

Partial Measurements for Qudits

Suppose Alice and Bob have entangled qutrits

$$\alpha_{11}|11\rangle + \alpha_{12}|12\rangle + \alpha_{11}|13\rangle + \alpha_{21}|21\rangle + \dots + \alpha_{33}|33\rangle$$

$$\mathbb{P}[\text{ Alice's measurement is "1" }] = |\alpha_{11}|^2 + |\alpha_{12}|^2 + |\alpha_{13}|^2 := p_1$$

& state becomes $\frac{\alpha_{11}|11\rangle + \alpha_{12}|12\rangle + \alpha_{13}|13\rangle}{\sqrt{p_1}}$

and so on ....

Fact to Remember : Measuring qubits one by one gives the same outcome as measuring both qubits ← Can do it in any order

Exercise (in-class) Give a circuit to prepare the Bell state starting from the state $|++\rangle$

EPR Paradox

Alice & Bob prepare the Bell state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

They each take one of the qubits & go far away

① If Alice measures her qubit in $\{|0\rangle, |1\rangle\}$ basis
Bob's qubit changes to whatever was measured

Is this faster-than-light communication ?

No! Alice & Bob learn a random bit

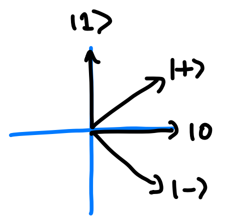
② But Alice learns Bob's outcome instanteously

Alice & Bob have two copies of a classical coin tossed by Charlie
They only look at it when they are far away
No violation of classical physics!

Local Hidden Variable

Local Hidden Variable Theory

③ Suppose Alice measures her qubit in a different basis
e.g. in $|\pm\rangle$ basis

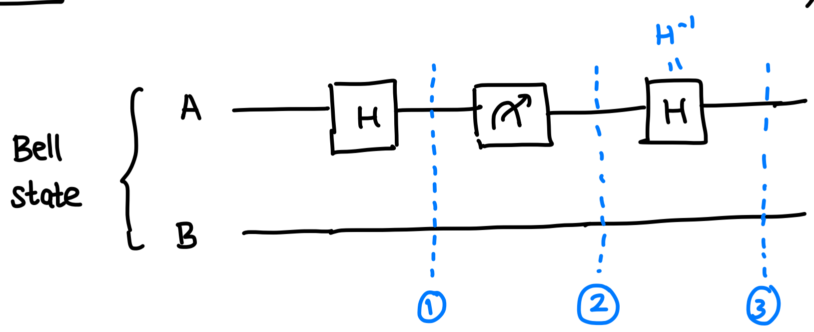


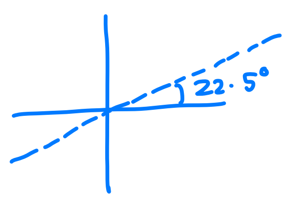
What happens to Bob's qubit ?

Let's compute it in two different ways

$\boxed{\text{First}}$ Simulate $|\pm\rangle$ measurement with unitary + std. basis measurement

H = reflection at 22.5°





$H: |0\rangle \to |+\rangle$
$\quad |1\rangle \to |-\rangle$

$H^{-1} = H: |+\rangle \to |0\rangle$
$\quad |-\rangle \to |1\rangle$

State ① :

$$\frac{1}{\sqrt{2}}(H|0\rangle)\otimes|0\rangle + \frac{1}{\sqrt{2}}(H|1\rangle)\otimes|1\rangle$$

$$= \frac{1}{\sqrt{2}}|+\rangle\otimes|0\rangle + \frac{1}{\sqrt{2}}|-\rangle\otimes|1\rangle$$

$$= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

State ②: A measures $\mathbb{P}[\text{ measures } 0] = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$

$\rightarrow$ state changes to $|0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = |0\rangle \otimes |+\rangle$

interprets as measuring "$|+\rangle$"

State ③: Rotate back to get the correct state when we measure in the $|\pm\rangle$ basis

$\rightarrow$ state becomes $|+\rangle \otimes |+\rangle$ $\rightarrow$ final state

Bob's state is also $|+\rangle$

In the other case, with probability $1/2$, measures "1" $\rightarrow$ interprets as "$|-\rangle$"

final state is $|-\rangle \otimes |-\rangle$

This is similar to what happened if Alice measured in $\{|0\rangle, |1\rangle\}$ basis

Here, with 50% chance, she either gets a $|+\rangle$ or a $|-\rangle$
& Bob's state collapses to whatever Alice measures

$\boxed{\text{Second}}$ let's do the above computation differently & directly try to measure in $|\pm\rangle$ basis

EPR pair : $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Let's express the first qubit in the $|\pm\rangle$ basis

$$|00\rangle = |0\rangle \otimes |0\rangle = \left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) \otimes |0\rangle$$

$$|11\rangle = |1\rangle \otimes |1\rangle = \left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right) \otimes |1\rangle$$

$$\text{Final state} = \frac{1}{2}|+0\rangle + \frac{1}{2}|-0\rangle + \frac{1}{2}|+1\rangle - \frac{1}{2}|-1\rangle$$

$$= |+\rangle \otimes \left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle\right) + |-\rangle \otimes \left(\frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle \quad !!!$$

EPR pair is an equal superposition of two different bases

If Alice measures in $|\pm\rangle$ basis :

$$\mathbb{P}[\text{measures "}|+\rangle\text{"}] = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$ and similarly for the other case

and state collapses to $|+\rangle$

This is more spooky than before because Alice can maybe convey some information to Bob instanteously by deciding to measure either in $\{|0\rangle, |1\rangle\}$ or $\{|+\rangle, |-\rangle\}$ basis

Bob's state changes to something that Alice knows which is different depending on the basis

Has Alice managed to convey one bit of information to Bob via the following protocol :

Alice wants to send "0" to Bob : Measure in std. basis

$$\left.\begin{array}{l} 50\% \text{ chance : Bob's state becomes } |0\rangle \\ 50\% \text{ chance : ——————— } |1\rangle \end{array}\right\} \text{Mixed state } \rho_0$$

Alice wants to send "1" to Bob : Measure in $\{|\pm\rangle\}$ basis

$$\left.\begin{array}{l} 50\% \text{ chance : Bob's state becomes } |+\rangle \\ 50\% \text{ chance : ——————— } |-\rangle \end{array}\right\} \text{Mixed state } \rho_1$$

Bob does some local operation on his qubit to decode the message

Resolution : There is no local operation that Bob can do that distinguishes the two mixed states ρ_0 & ρ_1

The question of whether quantum mechanics is a local hidden variable theory or not went unsolved for 30 years

Bell's Theorem No local hidden variable theory can be compatible with quantum mechanics

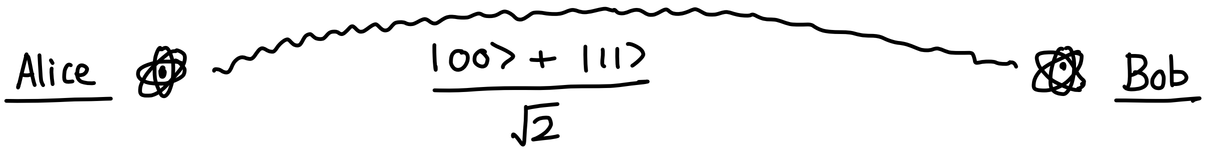
Bell in 1964 designed an experiment called the Bell test such that the predictions of quantum mechanics differ from the predictions of any local hidden variable theory

CHSH game was a simplification of Bell's experiment devised in 1970s by Clauser, Horne, Shimony and Holt ← One of the major discoveries in Quantum Mechanics !!

CHSH game

Alice & Bob prepare the Bell state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

They each take one of the qubits & go far away say Alice goes to Mars & Bob goes to Jupiter

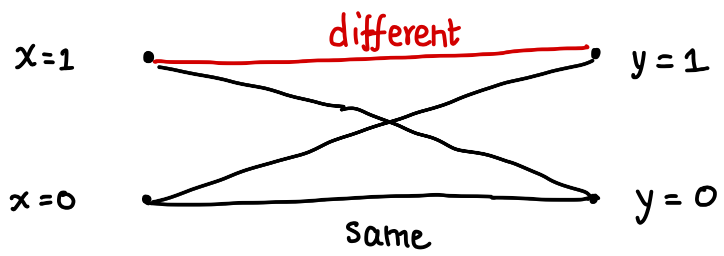


They are both issued a challenge by a referee as follows:

- Challenge to Alice is $x \in \{0,1\}$ and to Bob is $y \in \{0,1\}$ where x and y are independent random bits
- Referee puts the challenge in a box & Alice & Bob look at it at the same time
- They are both given 10 seconds to respond with a bit and Mars is at least 30 light minutes from Jupiter so no time for Alice to secretly communicate with Bob
- The boxes collect their responses and fly back to the referee
- They win the game if Alice's response bit $a \in \{0,1\}$ and Bob's response $b \in \{0,1\}$ satisfy the following

$$a \oplus b = x \wedge y$$

Another way of visualizing what happens in the game is via the following graph



Referee chooses a random edge and Alice & Bob's bit a & b should be different for the red edge and same otherwise in order to win the game

What is the maximum winning probability for Alice & Bob?

Deterministic Strategies Suppose Alice and Bob use deterministic strategies

Alice's answer a is a fixed function $a(x)$ of her question. Similarly, Bob's answer is also a function $b(y)$

For instance, Say Alice and Bob always answer 0

So, $a(x) = 0 \quad \forall x \in \{0,1\}$

$b(y) = 0 \quad \forall y \in \{0,1\}$

Then, they win if they get any of the black edges

$$\Rightarrow \quad \mathbb{P}[\text{winning}] = \frac{3}{4}$$

One can try all possible functions $a(x)$ and $b(y)$ and see that the maximum winning probability is $\frac{3}{4}$

Local Hidden Variable Strategy Suppose Alice & Bob are described by local hidden variables

This means that $\exists$ an underlying random variable λ such that

① Before the game, λ is sampled from some probability distribution $\mathcal{L}$

② Questions (x,y) sampled independently of λ

③ Alice's answer is a function $a(x,\lambda)$

④ Bob's answer is a function $b(y,\lambda)$

Note: One can think of λ as shared random coins

What is the maximum winning probability for Alice & Bob ?

The ability to use a hidden random variable λ does not help Alice & Bob : their maximum winning probability is $3/4$

$\boxed{\text{Proof}}$

$$\mathbb{P}[\text{win}] = \sum_{\lambda} \mathbb{P}[\lambda] \cdot \mathbb{P}[\text{win} \mid \lambda]$$

But if λ is fixed, Alice and Bob's answers are deterministic functions of their questions only meaning $\mathbb{P}[\text{win} \mid \lambda] \leq \frac{3}{4}$

Therefore, $\mathbb{P}[\text{win}] \leq \sum_{\lambda} \mathbb{P}[\lambda] \cdot \frac{3}{4} \leq \frac{3}{4}$

Einstein would have predicted that Alice & Bob cannot win with probability greater than $\frac{3}{4}$ in the CHSH game !!

What does Quantum Mechanics predict?

There exists a quantum strategy involving quantum entanglement where Alice & Bob win with probability $\approx 85\%$

This gives an experiment to rule out local hidden variable theories

<u>NEXT TIME</u> Quantum strategy for CHSH game & more