Homework 5

CS 498 MV: Logic in Computer Science

Assigned: November 29, 2018 Due on: December 7, 2018

Instructions: Solutions to the homework problems should be turned in as a PDF file on Gradescope. See instructions on Piazza.

Recommended Reading: Lectures 17 through 22: LTL, tree automata, finite model theory, descriptive complexity, and series-parallel graphs.

Homework Problems

Problem 1. Recall that in class we showed that connectivity is not expressible in ordered graphs using Gurevich's result about orders. In this problem you are required to show that it is not expressible in (unordered) graphs directly. Let G_ℓ denote a cycle of length $\ell + 1$. That is, G_ℓ has vertices $V = \{0, 1, \ldots \ell\}$ and edges $E = \{(i, i+1) \mid 0 \leq i < \ell\} \cup \{(i+1, i) \mid 0 \leq i < \ell\} \cup \{(0, \ell), (\ell, 0)\}$. Let $G_k + G_\ell$ denote the graph obtained by taking the disjoint union of G_k and G_ℓ, i.e., $G_k + G_\ell$ has two cycles of length $k+1$ and $\ell + 1$ over disjoint set of vertices. Show that the Duplicator has a winning strategy in $\mathcal{G}_m(G_\ell, G_k + G_\ell)$, when $k, \ell \geq 2^m$.

Problem 2. A *pushdown automaton* P (without inputs and final states) is a tuple $(Q, q_0, \Gamma, \perp, \delta)$, where Q is a finite set of states, $q_0 \in Q$ is the initial state, Γ is the finite stack alphabet, $\perp \notin \Gamma$ is the initial stack symbol, and $\delta = \delta_{\mathsf{push}} \cup \delta_{\mathsf{pop}} \cup \delta_{\mathsf{nochg}}$ is the transition relation. $\delta_{\mathsf{push}} \subseteq Q \times \Gamma \times Q$ are the push transitions: $(q_1, \gamma, q_2) \in \delta_{\mathsf{push}}$ means that the automaton can go from q_1 to q_2 by pushing γ onto the stack (independent of what is on top of the stack). $\delta_{\mathsf{pop}} \subseteq Q \times \Gamma \times Q$ are the pop transitions: $(q_1, \gamma, q_2) \in \delta_{\mathsf{pop}}$ means that the automaton can go from q_1 to q_2 if γ is on top of the stack, and the result of taking the transition is to pop γ from the stack. $\delta_{\mathsf{nochg}} \subseteq Q \times Q$ are the no stack change transitions: $(q_1, q_2) \in \delta_{\mathsf{nochg}}$ means that the automaton can go from q_1 to q_2 without changing the stack (independent of what is on top of the stack).

The *graph of a pushdown automaton* P, is the (infinite) graph $G(P) = (V, E)$, where $V = \perp\Gamma^* Q$ and E is defined as follows. $(\perp\gamma_1^1\gamma_2^1 \cdots \gamma_n^1 q_1, \perp\gamma_1^2\gamma_2^2 \cdots \gamma_m^2 q_2) \in E$ if one of the following 3 condtions hold. Either $(q_1, \gamma, q_2) \in \delta_{\mathsf{push}}$ and $m = n+1$, with $\gamma_m^2 = \gamma$ and $\gamma_i^2 = \gamma_i^1$ for $i \leq n$. Or $(q_1, \gamma, q_2) \in \delta_{\mathsf{pop}}$ and $m = n-1$, with $\gamma_n^1 = \gamma$ and $\gamma_i^2 = \gamma_i^1$ for $i \leq n-1$. Or $(q_1, q_2) \in \delta_{\mathsf{nochg}}$ and $m = n$, with $\gamma_i^2 = \gamma_i^1$ for $i \leq n$.

The *infinite n-ary tree* $\mathcal{T}$ is the infinite tree where every node has exactly n children. It can be viewed as a first order structure over the signature $\tau_n = \{S_1, S_2, \ldots S_n\}$, where S_i is the ith child relation. In other words, as a first order structure $\mathcal{T} = (T, \{S_i^{\mathcal{T}}\}_{i=1}^n)$, where T is the set of nodes, and $(u, v) \in S_i^{\mathcal{T}}$ iff v is the ith child of u.

Consider a pushdown automaton P with $\Gamma = \{1, 2, \ldots k\}$ and $Q = \{k+1, k+2, \ldots n\}$. Prove that for every MSO sentence φ (over signature $\tau_G = \{E\}$) there is an MSO sentence φ^* (over signature τ_n) such that

$$G(P) \models \varphi \qquad \text{iff} \qquad \mathcal{T} \models \varphi^*$$

Hint: Use the idea of interpretations. For a vertex $v = \perp\gamma_1\gamma_2 \cdots \gamma_m q$ of $G(P)$ let $t(v)$ be the node of $\mathcal{T}$ reached by following the path $\gamma_1\gamma_2 \cdots \gamma_m q$ from the root. And for a set of vertices U of $G(P)$, let $t(U) = \{t(v) \mid v \in U\}$. The translation to φ^* can be done inductively, maintaining the following invariant

$$G(P) \models \psi(x_1, \ldots, x_s, X_1, \ldots X_t)[x_i \mapsto v_i, X_i \mapsto U_i] \text{ iff } \mathcal{T} \models \psi^*(x_1, \ldots, x_s, X_1, \ldots, X_t)[x_i \mapsto t(v_i), X_i \mapsto t(U_i)]$$

In doing this construction, it might be useful to first define a formula $\mathrm{Vert}(x)$ such that for a node n_1 of $\mathcal{T}$ $\mathrm{Vert}(n_1)$ holds iff $n_1 = t(v)$ for some v of $G(P)$, i.e., n_1 is reached by following a path of the form $\gamma_1 \cdots \gamma_m q$, where $\gamma_i \in \{1, \dots k\}$ and $q \in \{k+1, \dots n\}$.