Homework 4
CS 498 MV: Logic in Computer Science

Assigned: October 29, 2018   Due on: November 9, 2018

Instructions: Please do not turn in solutions to the practice problems. Solutions to the homework problems should be turned in as a PDF file on Gradescope. See instructions on Piazza.

Recommended Reading: Lectures 12 through 16: MSO on words, decidable theories of arithmetic, and Büchi automata.

Practice Problems

Practice Problem 1. Give first order or MSO sentences describing the following languages

1. a^+b^*
2. aab^*aa
3. there are at least 3 occurrences of b and before the first occurrence of b there are at most 2 occurrences of a

Practice Problem 2. Construct Büchi automata that recognize the following languages over $\Sigma = \{a, b, c\}$.

1. The collection of all words where the substring abc appears at least once.
2. The collection of all words where the substring abc appears infinitely many times.
3. The collection of all words where the substring abc appears finitely many times.

Practice Problem 3. Prove or disprove the the following for $U, V \subseteq \Sigma^* \setminus \{\varepsilon\}$.

1. $(U \cup V)^\omega = U^\omega \cup V^\omega$
2. $\lim(U \cup V) = \lim U \cup \lim V$
3. $U^\omega = \lim U^+$, where $U^+ = U^* \setminus \{\varepsilon\}$
4. $\lim(U \cdot V) = U \cdot V^\omega$
5. $(U \cup V)^\omega = (U^*V)^\omega \cup (U \cup V)^*U^\omega$
6. $(U \cdot V)^\omega = U(VU)^\omega$

Practice Problem 4. Let A and B be languages recognized by deterministic Büchi automata. Prove that the following following languages are also recognized by deterministic Büchi automata.

1. $A \cup B$

2. $A \cap B$

Practice Problem 5. Let A be a Büchi recognizable language such that $A \neq \emptyset$. Prove that there are $u, v \in \Sigma^*$ such that $uv^\omega \in A$. That is, every nonempty Büchi recognizable language contains an *ultimately periodic* word (i.e., an infinite word of the form uv^ω).

Homework Problems

Problem 1. Recall that we showed that $\mathrm{Th}((\mathbb{N}, 0, +))$ is decidable. Show that the theory continues to be decidable when we add the following predicates.

1. $<$, a binary predicate, such that $<ij$ is true iff $i < j$
2. even, a unary predicate, such that $\mathsf{even}(i)$ iff i is even
3. $\mathsf{power2}$, a unary predicate, such that $\mathsf{power2}(i)$ iff i is a power of 2, or there is a j such that $i = 2^j$.

Sketch the proof, showing only the necessary constructions.

Problem 2. There are many ways to define real numbers. One way is to use *Dedekind cuts*. Intuitively, every real number $a \in \mathbb{R}$ can be expressed as a partition of rational numbers (S, T), where $S = \{s \in \mathbb{Q} \mid s < a\}$ and $T = \{t \in \mathbb{Q} \mid t \geq a\}$; since $T = \mathbb{Q} \setminus S$, we can think of the cut as just a single set S. Conversely, for any set S that is downward closed (i.e., $x < y$ and $y \in S$ implies $x \in S$) corresponds to the (unique) real number r that is the supremum of S. For example, $\sqrt{2}$ is represented by the set $S = \{s \in \mathbb{Q} | s^2 < 2 \textit{ or } s < 0\}$. Observe that $\sqrt{2} \notin S$ but $\sqrt{2} = \sup(S)$.

Assume that Dedekind cuts are a faithful representation of all real numbers (which they are). Using Dedekind cuts, show how to interpret first-order logic formulas over reals, i.e., over $(\mathbb{R}, +, <, 0, 1)$, into the *monadic second-order* theory over the rational numbers $(\mathbb{Q}, +, <, 0, 1)$. In other words, come up with a uniform way to map sentences FOL sentences φ over reals with addition to an MSOL sentence φ' over rationals with addition, such that φ holds over reals iff φ' holds over rationals.

Problem 3. Prove that the language $A = \{u^\omega \mid u \in \{0,1\}^* \setminus \{\varepsilon\}\}$ is not ω-regular. *Hint:* Consider words in the set $B_n = \{(01^k)^\omega \mid k \leq n\} \subseteq A$. Assuming A is ω-regular/Büchi recognizable, can you get a contradiction for an appropriate choice of n and B_n?