Homework 3
CS 498 MV: Logic in Computer Science

Assigned: October 8, 2018 Due on: October 16, 2018

Instructions: Please do not turn in solutions to the practice problems. Solutions to the homework problems should be turned in as a PDF file on Gradescope. See instructions on Piazza.

Recommended Reading: Lectures 3, 8, and 9: Compactness theorem and first order logic.

Practice Problems

Practice Problem 1. Express the following statements using predicates, quantifiers, logical connectives, and mathematical operators.

1. The product of two negative numbers is positive.
2. The difference of a real number and itself is zero.
3. Every positive real number has exactly two square roots.

Practice Problem 2. Show that the following sentences are not valid by constructing a structure in which they do not hold.

1. $(\forall x \exists y P(x, y)) \to (\exists y \forall x P(x, y))$
2. $\neg((\forall x \exists y P(x, y)) \to (\exists y \forall x P(x, y)))$
3. $(\forall x \exists y P(x, y)) \to (\exists y P(y, y))$

Practice Problem 3. Show that the names of bound variables are unimportant. In other words, show that $\forall x \varphi$ is logically equivalent to $\forall y \varphi^x_y$, where y is a fresh variable not appearing in φ.

Homework Problems

Problem 1. Recall that a set of sentences Γ is satisfiable, if there is a structure $\mathcal{A}$ such that for every $\varphi \in \Gamma$, $\mathcal{A} \models \varphi$. Further Γ is finitely satisfiable if every finite subset of Γ is satisfiable. Show that if a set of sentences Γ is finitely satisfiable then Γ is satisfiable. Assume that the signature τ being considered is finite. *Hint:* Follow the idea behind the proof using Henkin models of the compactness theorem for propositional logic (Section 3.2 of lecture 3 notes). That is, starting from Γ, construct a sequence $\Delta_0, \Delta_1, \ldots$ in the following manner. Let $c_1, c_2, \ldots$ be an infinite sequence of constant symbols not in τ, and let $\varphi_1, \varphi_2, \ldots$ be an enumeration of sentences over signature $\tau \cup \{c_1, c_2, \ldots\}$

- **Case 1:** If $\Delta_i \cup \{\varphi_{i+1}\}$ is finitely satisfiable then $\Delta_{i+1} = \Delta_i \cup \{\varphi_{i+1}\}$
- **Case 2:** If $\Delta_i \cup \{\varphi_{i+1}\}$ is not finitely satisfiable and φ_{i+1} is not of the form $\forall x \psi$ then $\Delta_{i+1} = \Delta_i \cup \{\neg\varphi_{i+1}\}$
- **Case 3:** If $\Delta_i \cup \{\varphi_{i+1}\}$ is not finitely satisfiable and φ_{i+1} is of the form $\forall x \psi$ then $\Delta_{i+1} = \Delta_i \cup \{\neg\varphi_{i+1}, \neg\psi^x_c\}$, where c is the first symbol in the list $c_1, c_2, \ldots$ not appearing in Δ_i

Show that Δ_i is finitely satisfiable for all i, and so is $\Delta = \bigcup_i \Delta_i$. Then conclude that the sentences in Δ can be used to construct a structure satisfying Δ and hence Γ.

Problem 2. The compactness theorem is evidence that first order logic is weak, from an expressive standpoint. Consider the signature of graphs $\tau_G = \{E\}$, where E is a binary relation. Prove that there is no sentence ψ over τ_G such that a structure $\mathcal{A} \models \psi$ if and only if $\mathcal{A}$ is finite. That is, you cannot express in first order logic that a structure has a finite universe.

Problem 3. Consider a signature τ that does not have any constant symbols. That is, τ only has relation symbols. Consider a τ-structure $\mathcal{A} = (A, \{R^{\mathcal{A}}\}_{R\in\tau})$. A structure $\mathcal{B} = (B, \{R^{\mathcal{B}}\}_{R\in\tau})$ is said to be a *submodel* of $\mathcal{A}$ if $B \subseteq A$ and for every $R \in \tau$, $R^{\mathcal{B}} = R^{\mathcal{A}} \cap B^k$, where k is the arity of R. In other words, the universe of $\mathcal{B}$ is a subset of $\mathcal{A}$ and every relation symbol has the same interpretation as $\mathcal{A}$ except that it is restricted to be over the tuples of B.

1. A sentence φ is said to be *universal* if it is of the form $\forall x_1 \forall x_2 \cdots \forall x_n \psi$, where ψ has no quantifiers. Prove that for any universal sentence φ, and two structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B}$ is a substructure of $\mathcal{A}$, if $\mathcal{A} \models \varphi$ then $\mathcal{B} \models \varphi$.
2. Let P be a unary relation symbol in τ. Using the previous part, prove that the sentence $\exists x P x$ is not equivalent to any universal formula.