Homework 2
CS 498 MV: Logic in Computer Science

Assigned: September 21, 2018 Due on: October 2, 2018

Instructions: Please do not turn in solutions to the practice problems. Solutions to the homework problems should be turned in as a PDF file on Gradescope. See instructions on Piazza.

Recommended Reading: Lectures 4 through 6: decidability, recursive enumerability, and complexity theory.

Practice Problems

Practice Problem 1. Prove the following properties about polynomial time reductions.

1. If $A \leq_{\mathrm{P}} B$ then $\overline{A} \leq_{\mathrm{P}} \overline{B}$.
2. If $A \leq_{\mathrm{P}} B$ and $B \leq_{\mathrm{P}} C$ then $A \leq_{\mathrm{P}} C$.

Practice Problem 2. Reductions allow one to compare the difficulty of two problems. But the notion of reduction needs to be tuned according to what one's goals are. This is illustrated in the following problem.

Consider the language $A = \{0\}$. Prove the following properties.

1. Let B be any decidable language. Then $B \leq_m A$.
2. Let $B \in \mathrm{P}$. Then $B \leq_{\mathrm{P}} A$.

This problem illustrates the deficiencies of different notions of reductions. Many-one reductions are too coarse to enable one to compare problems that are all decidable; this is evidenced by the fact that one can show the simple language A is "at least as hard" as every decidable language (with respect to many-one reductions). Similarly, polynomial time reduction are too coarse to compare problems that are in P.

Practice Problem 3. [Closure Properties] Consider languages $A, B \subseteq \Sigma^*$.

1. Prove that if $A, B \in \mathrm{P}$ then $A \cup B$, $A \cap B$, AB and $\overline{A}$ are in P.
2. Prove that if $A, B \in \mathrm{NP}$ then $A \cup B$, $A \cap B$, AB, and A^* are all in NP.

Practice Problem 4. Let $H = \{\langle i, x\rangle \mid M_i \text{ halts on } x\}$, where M_i is the Turing machine whose code is i. Prove that H is NP-hard. Is H NP-complete?

Homework Problems

Problem 1. Determine whether or not each of the following problems about Turing machines is decidable. Argue that your answer is correct. If your answer is "decidable", then describe an algorithm (that always halts) for the problem (at a suitably high level). If your answer is "undecidable", then prove your answer by constructing an appropriate reduction.

1. Given a TM M and input x, does M run for more than $2^{|x|}$ steps?
2. Given a TM M, is there an input x such that M runs for more than $2^{|x|}$ steps on x?
3. Given a TM M, is there an input x such that M runs for more than 2^n steps on x, where n is the number of states of M?
4. A computation is said to be nonerasing if, once a cell contains a nonblank character, it is never changed afterwards. Given a 2-tape TM M, and an input word x written on tape 1. Is the computation of M on x nonerasing?

Problem 2. Prove that the following problem, called MATCH is NP-complete. Given a finite set S of strings of length n over the alphabet $\{0, 1, *\}$ determine if there exists a string w of length n over the alphabet $\{0, 1\}$ such that for every string $s \in S$, s and w have the same symbol in at least one position. For example, if $S = \{001*, *100, 10*0, 1010\}$ then $w = 0000$ is a solution. However, if $S = \{00, *1, 1*\}$ then there is no string w that "matches".

Problem 3. Prove that if DSPACE(n) $\subseteq$ P then P = PSPACE. *Hint:* Let $L \subseteq \{0, 1\}^*$ be recognized by Turing machine M in space n^k. Define $L_{\text{pad}} = \{x\$^{|x|^k - |x|} \mid x \in L\}$. What can you say about space needed to solve L_{pad}?