Finite Model Theory

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Model theory is the study of mathematical structures like graphs, sets, algebras, et.c through the lens of logic. *Finite model theory* is the study of *finite* structures through logic. Before we study decision problems for and the expressive power of first order logic when resrticted to finite models, it is useful recall some classical results about first order logic that we saw before. This will help contrast the study of finite model theory when compared with classical model theory.

We begin by recalling Gödel's completeness theorem, which says that a sentence is valid if and only if it is provable. An immediate consequence of this observation is that the set of valid sentence is recursively enumerable; this is the Church-turing theorem.

Theorem 1 (Gödel, Church-Turing). $\Gamma \models \varphi$ *iff* $\Gamma \vdash \varphi$.
The set of valid (first-order) sentences is RE-complete.

Closely related to the completeness theorem is the *compactness theorem* which says that if a set of sentences is finitely satisfiable, then it is also satisfiable.

Theorem 2 (Compactness). Γ *is satisfiable iff* Γ *is finitely satisfiable.*

The compactness theorem demonstrates the expressive weakeness of first order logic. While it is possible to have sentences having only finite models, (example, $\forall x \forall y\, x = y$) it is not possible to have sentences all of whose models are finite, but which have models of arbitrary size. In other words, if a sentence has only finite models then there is a finite bound on the size of the models. This is the content of the following proposition.

Proposition 3. *If a set of sentences* Φ *has arbitrarily large finite models, then it has an infinite model.*

Proof. For integer $k \geq 2$ we can write a sentence $\mathsf{atleast}k$ which is satisfied by a structure if and only if it has at least k elements in its universe. This sentence can be written as

$$\mathsf{atleast}k = \exists x_1 \cdots \exists x_k (\wedge_{i \neq j} \neg(x_i = x_j))$$

Consider the set $\Gamma = \Phi \cup \{\mathsf{atleast}k \mid k \geq 2\}$. By the hypothesis of the propostion, every finite subset of Γ is satisfiable. Thus, by compactness, Γ is satisfiable. Now a structure satisfying Γ must have infinitely many elements, which means that Φ has an infinite model. □

Another important result in classical model theory, that can be seen as a consequence of the proof of the completeness theorem is the (Downward) Löwenheim Skolem Theorem; we have not seen this before, and its proof is beyond the scope of this course.

Theorem 4 ((Downward) Löwenheim-Skolem Theorem).

1. *Let* Φ *be a satisfiable set of sentences over a countable vocabulary (finite or countably infinite). Then* Φ *is satisfied in some countable structure.*
2. *If* Φ *is a satisfiable set of sentences over a vocabulary of cardinality* κ*, then* Φ *has a model of cardinality* $\leq \kappa$*.*

Example 5 (Skolem's Paradox)**.** Theorem 4 is surprising in the light of the following observation made by Skolem. Let Φ_{set} be some reasonable set of axioms for set theory. Since Φ_{set} is going to be satisfiable, by the above theorem, we can conclude that there is a countable structure $\mathcal{A}$ that satisfies Φ_{set}. Now the universe of $\mathcal{A}$ will consist of elements that represent the various sets, and $\mathcal{A}$ will also satisfy all the sentences that are logically implied by Φ_{set}. Now one consequences of Φ_{set} is a sentence, ψ, that (informally) says that there are uncountably many sets; essentially, the set of natural numbers is a set according to the axioms, and so are elements of its power set, which by Cantor's argument will imply the uncountability of the collection of all sets. However, $\mathcal{A}$ (which also satisfies this sentence) has only countably many elements! This is refered to as *Skolem's Paradox*, and is actually not a contradiction. The fact that $\mathcal{A}$ satisfies ψ only implies that there is no element in the universe of $\mathcal{A}$ that satisfies the formal definition of a one-to-one map of the natural numbers onto to the universe of $\mathcal{A}$. It does not in anyway exclude the possibility of there being (outside the universe of $\mathcal{A}$) some real function providing such a one-to-one correspondence.

The Löwenheim-Skolem Theorem gives us more evidence of the weakness of first-order logic's expressive power. But before presenting this application, a few definitions are in order.

Definition 6. For τ-structures $\mathcal{A}$ and $\mathcal{B}$, a function $h : u(\mathcal{A}) \to u(\mathcal{B})$ is called a *homomorphism* if it preserves relations and constants, i.e.,

- for every $c \in \tau$, $h(c^{\mathcal{A}}) = c^{\mathcal{B}}$,
- for every n-ary relation $R \in \tau$ and $a_1, \ldots a_n \in u(\mathcal{A})$, $(a_1 \ldots a_n) \in R^{\mathcal{A}}$ iff $(h(a_1) \ldots h(a_n)) \in R^{\mathcal{B}}$

A homomorphism h is an *isomorphism* if in addition h is one-to-one and onto. If there is an isomorphism $h : u(\mathcal{A}) \to u(\mathcal{B})$ then $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic* and is denoted by $\mathcal{A} \cong \mathcal{B}$.

Definition 7. Two structures $\mathcal{A}$ and $\mathcal{B}$ (of the same vocabulary τ) are said to be *elementarily equivalent*, written $\mathcal{A} \equiv \mathcal{B}$, if for every sentence φ (over τ), $\mathcal{A} \models \varphi$ if and only if $\mathcal{B} \models \varphi$. In other words, $\mathcal{A}$ and $\mathcal{B}$ satisfy exactly the same first-order sentences.

It follows from Definitions 6 and 7, that if $\mathcal{A} \cong \mathcal{B}$ then $\mathcal{A} \equiv \mathcal{B}$. A natural question to ask is if the converse holds; is it the case that there is always a first order sentence that will "distinguish" any two non-isomorphic structures? The answer is no because of Theorem 4.

Proposition 8. *There are structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \equiv \mathcal{B}$ but $\mathcal{A} \not\cong \mathcal{B}$.*

Proof. Consider $\mathcal{R} = (\mathbb{R}, \leq, 0, 1, +, \cdot)$, and $\Gamma = \text{Th}(\mathcal{R}) = \{\varphi \mid \mathcal{R} \models \varphi\}$. By Theorem 4, Γ has a countable model $\mathcal{A}$. Clearly, $\mathcal{A}$ is not isomorphic to $\mathcal{R}$, as $u(\mathcal{A})$ and $u(\mathcal{R})$ have different cardinalities. □

Compactness and the Downward Löwenheim-Skolem Theorem, together demonstrate that first order logic is not expressive enough to characterize models precisely. Since $\text{Th}(\mathcal{R})$ is an important first order theory, structures that have the same theory as $\text{Th}(\mathcal{R})$ have a special name.

Definition 9. $\mathcal{A}$ is a *real-closed field* if $\mathcal{A} \equiv (\mathbb{R}, \leq, 0, 1, +, \cdot)$.

1 Finite Model TheorY

We will now restrict our attention to the set of finite models of first order sentences. When we focus our attention to finite models, a number of observations we made before, no longer hold. In this section, we will investigate the same questions as before, but restrict our attention to finite models.

For finite models, both the completeness theorem and the compactness theorem fail. The failure of the Completeness Theorem is documented by Trakhtenbrot's Theorem.

Theorem 10 (Trakhtenbrot's Theorem)**.** *Given a sentence φ (over finite vocabulary τ), checking if φ is satisfiable in a finite model is RE-complete.*

Proof. The following non-deterministic Turing machine demonstrates membership in RE — guess the structure $\mathcal{A}$, and check if $\mathcal{A} \models \varphi$.

Hardness follows from the reduction in the proof of the Church-Turing theorem. □

Theorem 10 has immediate consequences to the validity problem over finite models.

Corollary 11. *Given a sentence φ (over finite vocabulary τ), checking if φ holds in all finite models is co-RE-complete. Thus, validity over finite models is not recursively enumerable.*

Corollary 11 means that the Completeness Theorem does not hold over finite models, i.e., there is no sound and complete proof system that reasons about finite models. The second pillar of classical model theory, namely the Compactness Theorem, also fails to hold when we focus on finite models.

Proposition 12. *There is a set of sentences Γ such that every finite subset of Γ has a finite model, but Γ itself, does not have a finite model,*

Proof. Recall the sentence,

$$\eta_{\geq k} = \exists x_1 \exists x_2 \cdots \exists x_k (\bigwedge_{i \neq j} \neg(x_i = x_j))$$

$\Gamma = \{\eta_{\geq k}\}_{k>0}$ satisfies the conditions of the proposition. □

Failure of the compactness theorem, suggests that another classical result, Theorem 4, is also unlikely to hold. In the context of finite models, we talk about small models. A logic $\mathcal{L}$ is said to have the *small model property* if there is a computable function f such that for every formula $\varphi \in \mathcal{L}$ if φ is satisfiable then there is a model of size $\leq f(|\varphi|)$ in which φ holds (where $|\varphi|$ is the size of φ). In other words, if a formula is satisfied then it is satisfied by a "small" model. A simple consequence of Trahtenbrot's Theorem is that first order logic does not have the small model property.

Corollary 13. *First order logic does have the small model property.*

Proof. Suppose first-order logic did have the small model property. Then there is a computable funtion f such that every satisfiable formula is satisfied in a structure of size bounded by the function f. This means that there is a decision procedure to check if a formula is satisfied in a finite model; the algorithm basically goes through all structures of size $\leq f(|\varphi|)$ and checks if φ holds in any of them. If none of these satisfy φ then we can conclude that φ is not satisfiable. This contradicts Trakhtenbrot's Theorem (Theorem 10) that says finite satisfiability is not decidable. □

The violation of the completeness theorem, the compactness theorem, and the small model property, seem to suggest that in some ways that first order logic restricted to finite models is very expressive. Another evidence of this is the following proposition.

Proposition 14. *For any finite structure $\mathcal{A}$, there is a sentence $\varphi_{\mathcal{A}}$ such that for any $\mathcal{B}$, if $\mathcal{B} \models \varphi_{\mathcal{A}}$ then $\mathcal{B} \cong \mathcal{A}$.*

Proof. Let τ be the signature and $u(\mathcal{A}) = \{a_1, \ldots a_n\}$. Then $\varphi_{\mathcal{A}}$ is

$$\begin{aligned} \exists x_1 \exists x_2 \cdots \exists x_n (\quad & \textstyle\bigwedge_{i \neq j} \neg(x_i = x_j) \wedge \forall x_{n+1} (\bigvee_i x_{n+1} = x_i) \wedge \\ & \textstyle\bigwedge_{c \in \tau,\ c^{\mathcal{A}} = a_i} (x_i = c) \wedge \\ & \textstyle\bigwedge_{R \in \tau,\ (a_{i_1}, \ldots a_{i_k}) \in R^{\mathcal{A}}} R x_{i_1} \cdots x_{i_k} \wedge \\ & \textstyle\bigwedge_{R \in \tau,\ (a_{i_1}, \ldots a_{i_k}) \notin R^{\mathcal{A}}} \neg R x_{i_1} \cdots x_{i_k} \end{aligned}$$

□

An immediate consequence of Proposition 14 is that elementary equivalence coincides with isomorphism for finite models.

Corollary 15. *If $\mathcal{A}$ and $\mathcal{B}$ are finite τ-structures and $\mathcal{A} \equiv \mathcal{B}$ then $\mathcal{A} \cong \mathcal{B}$.*

In addition, any set of finite models can be characterized exactly by a set of sentences.

Corollary 16. *Let K be any set of finite structures. There is a set of sentences Γ such that $[\![\Gamma]\!] = K$.*

Proof. Let $\eta_{\leq k} = \exists x_1 \cdots \exists x_k \forall x_{k+1} \vee_{j \leq k} (x_{k+1} = x_j)$, and $\eta_{=k} = \eta_{\leq k} \wedge \eta_{\geq k}$. Then

$$\Gamma = \{\eta_{=k} \to (\vee_{\mathcal{A} \in K,\ |\mathcal{A}|=k} \varphi_{\mathcal{A}}) \mid k \in \mathbb{N}\}$$

□

2 Expressive power of First Order Logic

Corollary 16 demonstrates the expressive power of first order logic when restricted to finite models — every collection of finite models can be defined using a *set of sentences*. In the section, we will explore when a set of finite models can be described by a *single* sentence (as opposed to a set of sentences). This leads us to the notion of definability which will be our subject of study.

Definition 17. A collection K of finite structures is *definable* if there is a sentence φ such that $[\![\varphi]\!] = K$.

We can come up with a simple characterization of when a collection of models is not definable. This characterization will then be exploited to characterize the expressive power of first order logic. In order to give this characterization, we introduce the notion of quantifier rank of formulas.

Definition 18. The *quantifier rank* of formula φ (denoted $\text{qr}(\varphi)$) is inductively defined as follows:

- $\text{qr}((t_i = t_j)) = \text{qr}(Rt_1 \cdots t_k) = 0$
- $\text{qr}(\neg\varphi) = \text{qr}(\varphi)$
- $\text{qr}(\varphi \vee \psi) = \max \text{qr}(\varphi), \text{qr}(\psi)$
- $\text{qr}(\exists x \varphi) = 1 + \text{qr}(\varphi)$

The quantifier rank of a formula simply counts the number of quantifiers that any subformula is in the scope of. Let us look at some examples. Atomic formulas have quantifier rank 0 — $\text{qr}(\neg(x = y)) = 0$. Recall that $\eta_{\leq k} = \exists x_1 \cdots \exists x_k \forall x_{k+1} \vee_{j \leq k} (x_{k+1} = x_j)$, and $\eta_{\geq k} = \neg\eta_{\leq k-1}$. We have $\text{qr}(\eta_{\leq k}) = k + 1$, $\text{qr}(\eta_{\geq k}) = k$, and $\text{qr}(\eta_{\leq k} \wedge \eta_{\geq k}) = k + 1$.

Distinguishing formulas based on quantifier rank, allows one to get a finer grained notion of elementary equivalence of structures.

Definition 19. $\mathcal{A}$ is *elementarily equivalent* to $\mathcal{B}$ *upto quantifier rank* m (denoted $\mathcal{A} \equiv_m \mathcal{B}$) iff for all φ with $\text{qr}(\varphi) \leq m$, $\mathcal{A} \models \varphi \leftrightarrow \mathcal{B} \models \varphi$.

Elementary equivalence upto a quantifier rank helps us get a simple characterization of non-definability.

Proposition 20. *If K is a collection of finite structures such that for every m, there exist finite structures $\mathcal{A}, \mathcal{B}$ such that*

- *$\mathcal{A} \equiv_m \mathcal{B}$, and*
- *$\mathcal{A} \in K$ and $\mathcal{B} \notin K$*

then K is not definable.

Proof. Suppose (for contradiction) K is definable by φ. Let $m_0 = \text{qr}(\varphi)$. Then, there exists $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \equiv_{m_0} \mathcal{B}$ but $\mathcal{A} \in K$ and $\mathcal{B} \notin K$. Since $\mathcal{A} \equiv_{m_0} \mathcal{B}$, we have $\mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$. But as $\mathcal{A} \in K$ and $\mathcal{B} \notin K$, φ cannot possibly define K as we know that either $\mathcal{A} \not\models \varphi$ or
$fosensat\mathcal{B}\varphi$. □

Is the converse of Proposition 20 true? That is, if for some m, K is a collection of equivalence classes of $\equiv_m$, then is K definable? The answer turns out to be yes. To establish this result, we will take a detour via games that help understand the expressive power of first order logic.

2.1 Ehrenheucht-Fraisse Games

We will now define a class of games, called Ehrenheucht-Fraisse games, that helps us characterize elementary equivalence upto quantifier rank m on finite structures. This characterization also helps understand the expressive power of first order logic on finite models. To define these games, we need to introduce the notion of *partial isomorphisms*, which are partial functions between structures that behave like on an isomorphism on their domain.

Recall that a partial function $f : A \hookrightarrow B$ is a mapping from A to B that may not be defined on all elements of A. The *domain* of the partial function f is the set of elements on which f is defined; we will denote it by $\mathsf{dom}(f)$. If $a \in \mathsf{dom}(f)$, will denote that sometimes by $f(a)\downarrow$. We will denote by $\emptyset$ the partial function whose domain is $\emptyset$.

Definition 21. A partial function $p : u(\mathcal{A}) \hookrightarrow u(\mathcal{B})$ is a *partial isomorphism* between τ-structures $\mathcal{A}$ and $\mathcal{B}$ iff

- p is 1-to-1,
- For every constant $c \in \tau$, $c^{\mathcal{A}} \in \mathsf{dom}(p)$ and $c^{\mathcal{B}} = p(c^{\mathcal{A}})$, and
- For every k-ary relation $R \in \tau$, and $a_1, \ldots a_k$ such that $a_i \in \mathsf{dom}(p)$, $(a_1, \ldots a_k) \in R^{\mathcal{A}}$ iff $(p(a_1), \ldots p(a_k)) \in R^{\mathcal{B}}$.

Observe that if p is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$, then p is also a partial isomorphism. Another simple example of a partial isomorphism is $\emptyset$, which is a partial isomorphism if the signature *tau* does not have any constant symbols. Partial isomorphism preserve the truth of *atomic formulas*; here we call φ an atomic formula if $\mathrm{qr}(\varphi) = 0$.

Proposition 22. *Let p be a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Let φ be an atomic formula with free variables $x_1, \ldots x_n$. Then, for every $a_1, \ldots a_n$ such that $a_i \in \mathsf{dom}(p)$,*

$$\mathcal{A} \models \varphi[[x_i \mapsto a_i]_{i=1}^n] \qquad \textit{iff} \qquad \mathcal{B} \models \varphi[[x_i \mapsto p(a_i)]_{i=1}^n]$$

Proof. Follows by induction on φ. □

Proposition 22 cannot be extended to formulas that are not atomic. This can be seen by the following example.

Example 23. Let $\mathcal{A} = (\{0,1,2,3\}, <^{\mathcal{A}})$ with $0 <^{\mathcal{A}} 1 <^{\mathcal{A}} 2 <^{\mathcal{A}}< 3$, and let $\mathcal{B} = (\{0,1,2\}, <^{\mathcal{B}})$ with $0 <^{\mathcal{B}} 1 <^{\mathcal{B}} 2$. Consider p such that $p(0) = 0$, $p(1) = 1$ and $p(3) = 2$. Let $\varphi(x,y) = \exists z(x < z \wedge z < y)$. Observe that $\mathcal{A} \models \varphi[[x \mapsto 1, y \mapsto 3]]$. However, $\mathcal{B} \not\models \varphi[[x \mapsto p(1) = 1, y \mapsto p(3) = 2]]$. Thus, satisfaction of non-atomic formulas is not preserved under partial isomorphisms.

The games we will introduce, strengthen partial isomorphism to help characterize the preservation of non-atomic formulas. To define these games, it will be convenient to introduce some notation for describing partial isomorphisms. Let $a_1, a_2, \ldots, a_n \in u(\mathcal{A})$ and $b_1, b_2, \ldots, b_n \in u(\mathcal{B})$. then $p = \{a_i \mapsto b_i\}_{i=1}^n$ denotes the partial isomorphism where $p(c^{\mathcal{A}}) = c^{\mathcal{B}}$ for every constant $c \in \tau$, and $p(a_i) = b_i$; we will assume that if $a_i = c^{\mathcal{A}}$ for any $c \in \tau$, then $b_i = c^{\mathcal{B}}$.

Definition 24 (E-F Games)**.** The m-round Ehrenfeucht-Fraisse games (E-F game, for short) on structures $\mathcal{A}$ and $\mathcal{B}$ (denoted $G_m(\mathcal{A}, \mathcal{B})$) is a game between two players, Spoiler (S) and Duplicator (D) where in each round i,

- S chooses a structure (either $\mathcal{A}$ or $\mathcal{B}$) and picks an element from the chosen structure (either $a_i \in u(\mathcal{A})$ or $b_i \in u(\mathcal{B})$), and
- D then responds by picking an element from the other structure (either $b_i \in u(\mathcal{B})$ or $a_i \in u(\mathcal{A})$).

Example 25. $\mathcal{A}$: $a \longrightarrow b \longrightarrow c \longrightarrow d$ $\quad$ $\mathcal{B}$: $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$

Figure 1: Two order structures $\mathcal{A}$ and $\mathcal{B}$ are shown. $i \to j$ means that i is less than j.

$\mathcal{A}:$ $a \longrightarrow b \longrightarrow c \longrightarrow d \longrightarrow e \longrightarrow f \longrightarrow g \longrightarrow h \longrightarrow i$

$\mathcal{B}:$ $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6 \longrightarrow 7 \longrightarrow 8$

Figure 2: Two order structures $\mathcal{A}$ and $\mathcal{B}$ are shown. $i \to j$ means that i is less than j.

Duplicator wins the play $(a_1, b_1), (a_2, b_2), \ldots (a_m, b_m)$ if $\{a_i \mapsto b_i\}_{i=1}^{m}$ is a partial isomorphism.

The Duplicator (Spoiler) wins the game $G_m(\mathcal{A}, \mathcal{B})$ if there is a strategy for the Duplicator (Spoiler) such that no matter how the Spoiler (Duplicator) plays, the Duplicator (Spoiler) wins the game.

Let us look at some examples to help understand E-F games. We will often use D for the Duplicator, and S for the Spoiler when describing these games.

Consider the ordered structures $\mathcal{A} = (\{a, b, c, d\}, \leq)$ and $\mathcal{B} = (\{1, 2, 3, 4, 5\}, \leq)$ shown in Figure 1. We can argue that the Spoiler wins $G_3(\mathcal{A}, \mathcal{B})$ as follows.

- 1st round: S picks 3. D has to pick b or c.
- 2nd round: If D picks b, then S picks 1, and if D picks c then S picks 5. Let us consider the case when S picks 1. Now D has to pick a.
- 3rd round: Now S picks 2, and D is stuck.

Let us instead consider the structures $\mathcal{A} = (\{a, b, c, d, e, f, g, h, i\}, \leq)$ and $\mathcal{B} = (\{1, 2, 3, 4, 5, 6, 7, 8\}, \leq)$ shown in Figure 2. Now the Duplicator wins $G_3(\mathcal{A}, \mathcal{B})$. We will give a general strategy for the Duplicator to win in such games in Theorem 28.

Let us now make some simple observations about such games. First observe that if the game is played on isomorphic structures then the Duplicator can always play in manner that she wins.

Proposition 26. *If $\mathcal{A} \cong \mathcal{B}$ then the duplicator wins $G_m(\mathcal{A}, \mathcal{B})$ for any m.*

Proof. Let $f : u(\mathcal{A}) \to u(\mathcal{B})$ be an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. The strategy for the duplicator is as follows: whenever S picks $a \in u(\mathcal{A})$, then D picks $f(a)$, and when S picks $b \in u(\mathcal{B})$, D picks $f^{-1}(b)$. $\square$

Another straightforward observation is that if the Duplicator wins a game with m rounds, then she also wins the game is it is played for fewer than m rounds. Dually, if the Spoiler wins the m-round game then she wins games played for more than m-rounds as well.

Proposition 27. *If the Duplicator wins $G_m(\mathcal{A}, \mathcal{B})$ then the Duplicator also wins $G_n(\mathcal{A}, \mathcal{B})$, where $n \leq m$. If the Spoiler wins $G_m(\mathcal{A}, \mathcal{B})$ then the Spoiler also wins $G_n(\mathcal{A}, \mathcal{B})$, where $n \geq m$.*

While Proposition 26 identifies a simple case when the Duplicator can win, there are contexts where the structures are not isomorphic but the Duplicator can nonetheless win. This case was identified in Example 25. The structures we will consider are ordered structures. What we mean is that the signature is $\tau_O = \{\leq\}$ and the structures we consider interpret $\leq$ as an ordering relation on the universe. On such ordered structures we have the following observation due to Gurevich.

Theorem 28 (Gurevich)**.** *Consider structures $\mathcal{A} = (A, \leq)$ and $\mathcal{B} = (B, \leq)$. If $|A| \geq 2^m$ and $|B| \geq 2^m$ then the duplicator wins $G_m(\mathcal{A}, \mathcal{B})$.*

Proof. Let a_{min} and a_{max} be the smallest and largest elements in $\mathcal{A}$, and let b_{min} and b_{max} be the smallest and largest elements in $\mathcal{B}$. The result is proved by induction on m. In the base case, when $m = 0$, the observation holds trivially. Let us consider the game $G_m(\mathcal{A}, \mathcal{B})$ in the induction step. Without loss of generality, suppose S picks $a \in u(\mathcal{A})$. There are 2 cases to consider based on a's relationship with a_{min} and a_{max}.

- *Case* $|a - a_{\mathsf{min}}| < 2^{m-1}$. Then $|a_{\mathsf{max}} - a| > 2^{m-1}$. D will pick b such that $|b - b_{\mathsf{min}}| = |a - a_{\mathsf{min}}|$. In subsequent rounds, when S picks in $[a, a_{\mathsf{min}}]$ (or $[b_{\mathsf{min}}, b]$) D will an element in $[b_{\mathsf{min}}, b]$ ($[a_{\mathsf{min}}, a]$) according to the isomorphism, and when S picks in $[a, a_{\mathsf{max}}]$ (or $[b, b_{\mathsf{max}}]$) D will pick from $[b, b_{\mathsf{max}}]$ (or $[a, a_{\mathsf{max}}]$) according to the winning strategy constructed by the induction hypothesis.
- *Case* $|a_{\mathsf{max}} - a| < 2^{m-1}$. D picks b such that $|b_{\mathsf{max}} - b| = |a_{\mathsf{max}} - a|$. Correctness as in previous case.
- *Otherwise,* D picks b such that $|b - b_{\mathsf{min}}|$ and $|b_{\mathsf{max}} - b|$ are both $\geq 2^{m-1}$. Correctness follows from the strategy constructed inductively.

□

Theorem 28 identifies a situation when the Duplicator can even on structures that are not isomorphic. However, if the game played for enough rounds, then the Duplicator can only win if the structures are isomorphic.

Proposition 29. *If* $|u(\mathcal{A})| = m$ *and duplicator wins* $G_{m+1}(\mathcal{A}, \mathcal{B})$ *then* $\mathcal{A} \cong \mathcal{B}$.

Proof. Suppose $\mathcal{A} \not\cong \mathcal{B}$. We consider two cases. Let us first consider the case when $|u(\mathcal{A})| = |u(\mathcal{B})|$. Then, there is no 1-to-1 onto function that is an isomorphism. The Spoiler's strategy is to pick a new element in each round, and after m rounds the game constructs a bijective function. Since the constructed bijective function is not an isomorphism (and hence not a partial isomorphism), S wins in m rounds.

On the other hand, suppose $|u(\mathcal{A})| \neq |u(\mathcal{B})|$. Again S picks new element in each round from larger structure. After as many rounds as smaller structure (which is $\leq m$), S can pick a new element from the larger structure and D will lose. □

E-F games allow us prove the elementary equivalence of structures for sentences upto a certain quantifier rank, provided the Duplicator can win. In order to prove such a result, it is convenient to consider a generalization of the E-F game that we have looked at so far — instead of starting with no elements picked, we start the game when some elements of each structure have already been identified. Let us define this precisely.

Definition 30. Let $\overrightarrow{a} = a_1, \ldots a_s$ and $\overrightarrow{b} = b_1, \ldots b_s$. The m-round E-F game on structures $\mathcal{A}$ and $\mathcal{B}$ with pre-picked positions $\overrightarrow{a}$ and $\overrightarrow{b}$ (denoted $G_m(\mathcal{A}, \overrightarrow{a}, \mathcal{B}, \overrightarrow{b})$) proceeds like $G_m(\mathcal{A}, \mathcal{B})$ and the Duplicator wins if $\overrightarrow{a}\overrightarrow{a'} \mapsto \overrightarrow{b}\overrightarrow{b'}$ is a partial isomorphism, where

- $\overrightarrow{a'} = a'_1, \ldots a'_m$ are the elements of $\mathcal{A}$ picked during the game and $\overrightarrow{b'} = b'_1, \ldots b'_m$ are the elements of $\mathcal{B}$ picked during the game, and
- $\overrightarrow{a}\overrightarrow{a'} \mapsto \overrightarrow{b}\overrightarrow{b'}$ denotes the partial isomorphism $\{a_i \mapsto b_i, a'_j \mapsto b'_j\}_{i=1,j+1}^{i=s,j=m}$.

Given the definition of E-F games with pre-picked positions, it is easy to argue that the following two properties hold. First, if D wins $G_0(\mathcal{A}, \overrightarrow{a}, \mathcal{B}, \overrightarrow{b})$ then for any atomic formula $\varphi(x_1, \ldots x_s)$,

$$\mathcal{A} \models \varphi[[x_i \mapsto a_i]_{i=1}^s] \qquad \text{iff} \qquad \mathcal{B} \models \varphi[[x_i \mapsto b_i]_{i=1}^s].$$

Second, D wins $G_m(\mathcal{A}, \overrightarrow{a}, \mathcal{B}, \overrightarrow{b})$ iff for every $a' \in u(\mathcal{A})$ there is a $b' \in u(\mathcal{B})$ such that D wins $G_{m-1}(\mathcal{A}, \overrightarrow{a}a', \mathcal{B}, \overrightarrow{b}b')$ and for every $b' \in u(\mathcal{B})$ there is $a' \in u(\mathcal{A})$ such that D wins $G_{m-1}(\mathcal{A}, \overrightarrow{a}a', \mathcal{B}, \overrightarrow{b}b')$. This suggests that we could characterize when the Duplicator can win a game by a single formula. These formulas are called *Scott-Hintikka* formulas.

Definition 31. For a structure $\mathcal{A}$ and $\overrightarrow{a} = a_1, a_2, \ldots a_s$ define $\varphi^m_{\mathcal{A},\overrightarrow{a}}$ inductively on m as follows.

$$\begin{aligned}\varphi^0_{\mathcal{A},\overrightarrow{a}} &= \bigwedge_{\varphi \text{ is atomic, } \mathcal{A}\models\varphi[[x_i\mapsto a_i]]} \varphi(x_1, \ldots x_s)\\ \varphi^m_{\mathcal{A},\overrightarrow{a}} &= \bigwedge_{a'\in u(\mathcal{A})} \exists x_{s+1}\varphi^{m-1}_{\mathcal{A},\overrightarrow{a}a'} \wedge \forall x_{s+1} \bigvee_{a'\in u(\mathcal{A})} \varphi^{m-1}_{\mathcal{A},\overrightarrow{a}a'}\end{aligned}$$

The Scott-Hintikka formulas have quantifier rank m, and for a finite structure $\mathcal{A}$, there are only finitely many Scott-Hintikka formulas (of rank m). Both these observations can be established by induction on m.

Proposition 32. *The properties hold of Scott-Hintikka formulas. First, $qr(\varphi^m_{\mathcal{A},\overrightarrow{a}}) = m$. Second, for any s, m $\{\varphi^m_{\mathcal{A},\overrightarrow{a}} \mid \overrightarrow{a} \in u(\mathcal{A})^s\}$ is finite.*

Using Scott-Hintikka formulas we can characterize the conditions when the Duplicator can win an E-F game with pre-picked positions.

Theorem 33 (Ehrenfeucht)**.** *The following are equivalent.*

(a) D wins $G_m(\mathcal{A}, \overrightarrow{a}, \mathcal{B}, \overrightarrow{b})$

(b) $\mathcal{B} \models \varphi^m_{\mathcal{A},\overrightarrow{a}}[[x_i \mapsto b_i]^s_{i=1}]$

(c) For every $\varphi(x_1, \ldots x_s)$ with $qr(\varphi) \leq m$

$$\mathcal{A} \models \varphi[[x_i \mapsto a_i]^m_{i=1}] \qquad \textit{iff} \qquad \mathcal{B} \models \varphi[[x_i \mapsto b_i]^m_{i=1}]$$

Proof. It is easy to see that (c) ⇒ (b). To complete the proof, we will establish that (b) ⇒ (a), and (a) ⇒ (c).

((b) ⇒ (a)) We will prove this by induction on m. The base case of $m = 0$ follows immediate from properties of partial isomorphisms. For the induction step, observe that D wins $G_m(\mathcal{A}, \overrightarrow{a}, \mathcal{B}, \overrightarrow{b})$ iff for every $a' \in u(\mathcal{A})$ there is $b' \in u(\mathcal{B})$ such that D wins $G_{m-1}(\mathcal{A}, \overrightarrow{a}a', \mathcal{B}, \overrightarrow{b}b')$ and for every $b' \in u(\mathcal{B})$ there is $a' \in u(\mathcal{A})$ such that D wins $G_{m-1}(\mathcal{A}, \overrightarrow{a}a', \mathcal{B}, \overrightarrow{b}b')$ iff (by induction hypothesis) for every $a' \in u(\mathcal{A})$ there is $b' \in u(\mathcal{B})$ such that $\mathcal{B} \models \varphi^{m-1}_{\mathcal{A},\overrightarrow{a}a'}[[x_1 \mapsto b_1, \ldots x_s \mapsto b_s, x_{s+1} \mapsto b']]$ and for every $b' \in u(\mathcal{B})$ there is $a' \in u(\mathcal{A})$ such that $\mathcal{B} \models \varphi^{m-1}_{\mathcal{A},\overrightarrow{a}a'}[[x_1 \mapsto b_1, \ldots x_s \mapsto b_s, x_{s+1} \mapsto b']]$ iff

$$\mathcal{B} \models \bigwedge_{a'\in u(\mathcal{A})} \exists x_{s+1}\varphi^{m-1}_{\mathcal{A},\overrightarrow{a}a'} \wedge \forall x_{s+1} \bigvee_{a'\in u(\mathcal{A})} \varphi^{m-1}_{\mathcal{A},\overrightarrow{a}a'}[[x_i \mapsto b_i]^m_{i=1}]$$

((a) ⇒ (c)) Again we will prove by induction on m. The base case is straightforward. For the induction step, consider $\varphi = \exists x\psi$, where $\text{qr}(\psi) \leq m - 1$. Now $\mathcal{A} \models \exists x\psi[[x_i \mapsto a_i]^s_{i=1}]$ iff there is $a' \in u(\mathcal{A})$ such that $\mathcal{A} \models \psi[[x_1 \mapsto a_1, \ldots x_s \mapsto a_s, x \mapsto a']]$. Since D wins the game, there $b' \in u(\mathcal{B})$ such that D wins $G_{m-1}(\mathcal{A}, \overrightarrow{a}a', \mathcal{B}, \overrightarrow{b}, b')$. By induction hypothesis, $\mathcal{B} \models \psi[[x_1 \mapsto b_1, \ldots x_s \mapsto b_s, x \mapsto b']]$ which means $\mathcal{B} \models \exists x\psi[[x_i \mapsto b_i]^m_{i=1}]$. □

Let us conclude this section by showing that the converse of Proposition 20 holds.

Theorem 34. *Let K be a set of finite structures. K is not definable iff for every m, there are $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \equiv_m \mathcal{B}$ and $\mathcal{A} \in K$ and $\mathcal{B} \notin K$.*

Proof. (⇐) This is Proposition 20 that we have already proved.

(⇒) We will prove the contrapositive. Suppose for some m, for every $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \equiv_m \mathcal{B}$ implies $\mathcal{A} \in K$ iff $\mathcal{B} \in K$. Take $\varphi = \vee_{\mathcal{A}\in K}\varphi^m_{\mathcal{A}}$. We will show that φ defines K. If $\mathcal{B} \in K$ then $\varphi^m_{\mathcal{B}} \models \varphi$ and so $\mathcal{B} \models \varphi$. On the other hand, if $\mathcal{B} \models \varphi$ then $\mathcal{B} \models \varphi^m_{\mathcal{A}}$ for some $\mathcal{A} \in K$. From properties of Scott-Hintikka formulas, we have $\mathcal{B} \equiv_m \mathcal{A}$, which means $\mathcal{B} \in K$. □

2.2 Inexpressivity Results

The results established in the previous section allows one to demonstrate that, eventhough the compactness theorem does not hold, first order logic cannot express certain simple properties over finite structures. Since compactness does not hold, our inexpressivity results in the finite case rely on Theorem 33. Our first observation is that "evenness" cannot be expressed in first order logic. We saw this observation in the context of first order logic restricted to word structures, when we showed that checking the parity of some symbol (like evenness) is a counting property and that first order logic can only express non-counting properties. Here, we present a different argument.

Proposition 35. *Let $\tau_O = \{\leq\}$ be the signature of linear orders. There is no sentence φ over τ such that $(A, \leq) \models \varphi$ iff $|A|$ is even.*

Proof. Suppose (for contradiction) φ_{even} expresses the desired property. Let $\text{qr}(\varphi_{\text{even}}) = m$. Consider $\mathcal{A}$ and $\mathcal{B}$ such that $|u(\mathcal{A})| = 2^m$ and $|u(\mathcal{B})| = 2^m + 1$. By Theorem 28, D wins $G_m(\mathcal{A}, \mathcal{B})$. By Theorem 33, $\mathcal{A} \equiv_m \mathcal{B}$. Since $\mathcal{A} \models \varphi_{\text{even}}$ implies $\mathcal{B} \models \varphi_{\text{even}}$, which contradicts the fact that φ_{even} expresses evenness. □

We can use Proposition 35 to also show that another natural property on graphs is not expressible in first order logic. "Connectedness" is the property of (undirected) graphs, where every vertex has a path to every other vertex. We will prove this result for *ordered graphs*. Ordered graphs are graphs where there is an ordering relation on the vertices; the ordering itself has not correlation with the edge relation on the graph. Thus, ordered graphs are structures over the signature $\tau_{OG} = \{\leq, E\}$, where $\leq$ is an ordering relation on the universe, and E is a symmetric binary relation.

Proposition 36. *Let $\tau_{OG} = \{\leq, E\}$ be the signature of ordered graphs. There is no sentence φ over τ_{OG} such that $G \models \varphi$ iff G is connected.*

Proof. The idea is to show that if connectedness were expressible, then even-ness would also be expressible. The proof goes as follows. Given a linear order $\mathcal{A} = (\{1, 2, \ldots n\}, \leq^{\mathcal{A}})$, we will construct a graph $G(\mathcal{A})$ such that $G(\mathcal{A})$ is connected iff $\mathcal{A}$ is odd; here we assume that $\leq^{\mathcal{A}}$ is the standard ordering relation on the set $\{1, 2, \ldots n\}$. The idea behind $G(\mathcal{A})$ is as follows. The vertices will be $\{1, \ldots n\}$. 1 has an edge $n - 1$ and 2 has an edge to n. In addition, for every i, i has an edge to $i + 2$.

Suppose φ_c expressed connectivity. Consider the following formula

$$\varphi_E(x, y) = \quad (\mathsf{first}(x) \wedge \mathsf{second} - \mathsf{last}(y)) \vee (\mathsf{second}(x) \wedge \mathsf{last}(y)) \vee (``y = x + 2'').$$

Consider $\varphi_o = \varphi_c[Exy \mapsto \varphi_E(x, y)]$. Observe,

$$\mathcal{A} \models \varphi_o \qquad \text{iff} \qquad |u(\mathcal{A})| \text{ is odd}$$

□