CS 498mp3: Logic in Computer Science Spring 2017: Homework 2
Thursday 9:30am, March 16th
Hand over in class, before lecture begins.

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1. **Inexpressiveness of FOL using compactness theorem** [20 points]

   Fix a first order signature with a single binary relation E. You can assume E is symmetric, i.e., $\forall x \forall y. E(x, y) \leftrightarrow E(x, y)$ holds. Hence the models can be seen as (finite or infinite) undirected graphs.

   Prove that there is no set of FOL formulas X such that X holds precisely in the class of graphs that have finite degree.

   (A graph has finite degree if every vertex has only a finite number of neighbors.)

2. **Model construction** [20 points]

   Let us assume that we have FOL *without* equality.

   Let us assume a signature with one binary relation R, one binary function f, and a binary relation $\equiv$.

   A model is said to be a *normal* model if $\equiv$ is interpreted in the model as the identity relation. I.e., $u \equiv v$ *iff* $u = v$.

   Now let us try to add axioms that demand that $\equiv$ is an equivalence relation (in fact, a congruence) using the following formula ψ:

$$\forall x \quad x \equiv x \quad \wedge$$

$$\forall x \; \forall y \quad (x \equiv y \leftrightarrow y \equiv x) \quad \wedge$$

$$\forall x \; \forall y \; \forall z \quad (\,(x \equiv y \wedge y \equiv z) \Rightarrow x \equiv z) \quad \wedge$$

$$\forall x \; \forall y \; \forall z \quad (\,(x \equiv y \wedge y \equiv z) \Rightarrow x \equiv z) \quad \wedge$$

$$\forall x \forall x' \; \forall y \forall y' \quad (x \equiv x' \wedge y \equiv y') \Rightarrow (R(x, y) \Leftrightarrow R(x', y')) \quad \wedge$$

$$\forall x \forall x' \; \forall y \forall y' \quad (x \equiv x' \wedge y \equiv y') \Rightarrow (f(x, y) \equiv f(x', y')) \quad \wedge$$

   (a) For any model M that satisfies ψ, prove that there is a normal model M' such that for every formula φ in FOL, $M \models \varphi$ iff $M' \models \varphi$.

   You must prove the above formally by constructing M' and then arguing, by induction, that $M \models \varphi$ iff $M' \models \varphi$, for every formula φ.

   Note: The above can be easily extended to arbitrary signatures, and shows that the axiom ψ captures equality well enough that FOL cannot distinguish between true equality and a congruence relation.

(b) Assume that ψ' is the above formula ψ with the last conjunct omitted. Give a concrete formula φ that is satisfiable but does not hold in any normal model. You need to give a satisfying model and a brief argument as to why no normal model will satisfy your formula.

3. **Understanding FOL** [15 points]

   Show that none of the following sentences is logically implied by the conjunction of the other two.
   Do this by showing for every sentence a model such that the sentence is false in the model but where the other two sentences are true.

   (a) $\forall x\ \forall y\ \forall z\ (P(x, y) \Rightarrow P(y, z) \Rightarrow P(x, z))$

   (b) $\forall x\ \forall y\ (P(x, y) \Rightarrow P(y, x) \Rightarrow x = y)$

   (c) $(\forall x\ \exists y\ P(x, y)) \Rightarrow (\exists y\ \forall x\ P(x, y))$

4. **Inexpressiveness** [15 points]

   Assume that a signature contains relation symbols but no function symbols or constants. A model $\mathcal{N}$ is said to be a submodel of a model $\mathcal{M}$ if the universe of $\mathcal{N}$ is a subset of the universe of $\mathcal{M}$ and the interpretation of every relation in $\mathcal{N}$ is precisely the interpretation of the relation in $\mathcal{M}$ but restricted to tuples over $\mathcal{N}$.

   A *universal formula* is a formula of the form $\forall x_1\ \forall x_2 \ldots \forall x_n \alpha$ where α is a quantifier-free formula using only the variables $\{x_1, \ldots x_n\}$.

   (a) Show that for any two models $\mathcal{M}$ and $\mathcal{N}$, where $\mathcal{N}$ is a submodel of $\mathcal{M}$, then for any universal formula φ, if $\mathcal{M} \models \varphi$ then $\mathcal{N} \models \varphi$.

   (b) Using the above, prove that the formula $\exists x P(x)$ is not equivalent to *any* universal formula.

5. **Modeling Nash equilibria first-order logic over reals** [30 points]

   (You may want to read a bit about Nash equilibria.)

   Let's assume that two players S and R play tennis, with S serving to receiver R. S must choose to serve forehand (F) or backhand (B), and player R can expect that the serve will be a forehand (F) or a backhand. The following depicts the pay-off for player R and player S, respectively.

		Server S	
		F	B
Receiver R	F	90, 10	20, 80
	B	30, 70	60, 40

   If R receives the serve she expected, her payoff is greater. If it's the opposite of what she expected, then her payoff is less. (For example, if she expects forehand and gets a forehand, then payoff is 90 while if she gets a backhand, her payoff is 20.)

Nash's theorem says that this game (and any such game) has a Nash equilibrium in terms of a mixed strategy for the players. More precisely, there is an r and s such that player S plays F with probability s (and B with probability $1 - s$) and player R expects F with probability r (and B with probability $1-r$) such that no player has any incentive to deviate. In other words, with R's strategy fixed (using r), no matter how S plays (with another probability), S's expectation will not increase. And similarly, if S's strategy is fixed (using s), no matter how R plays (with another probability), R's expectation will not increase.

(a) Write a first order logic formula $\varphi(r, s)$ over the reals (with addition and multiplication) that expresses that r and s form a Nash equilibrium (and that $\exists r, s\varphi$ expresses the existence of a Nash equilibrium for the above game, which by Nash's theorem is a valid formula in real arithmetic.)

(b) Express the above in the SMT solver Z3 to find a mixed strategy for both players in a Nash equilibrium. Note that this involved *quantifiers* and though the standard formulation in Z3 should work, I cannot be absolutely sure! But do try, and in case it doesn't work, it's okay to give the precise formulation in Z3 and Z3's output saying it was unable to solve it (if it says UNSAT, there's something wrong with your formulation!)

Aside: The above can be easily modified for arbitrary two-player two-action games, and we can express that Nash equilibria always exist for any game-matrix using FOL over reals, and hence can be decided as FOL over reals is a decidable theory!