Heavy Hitters

Lecture 08

September 15, 2022

Models

Richer model:

- Want to estimate a function of a vector $x \in \mathbb{R}^n$ which is initially assume to be the all 0's vector.
- Each element e_j of a stream is a tuple (i_j, Δ_j) where $i_j \in [n]$ and $\Delta_i \in \mathbb{R}$ is a real-value: this updates x_{i_j} to $x_{i_j} + \Delta_j$. (Δ_j can be positive or negative)

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- $\Delta_j > 0$: *cash register* model. Special case is $\Delta_j = 1$.
- Δ_j arbitrary: *turnstile* model
- Δ_j arbitrary but $x \geq 0$ at all times: *strict turnstile* model
- *Sliding window* model: interested only in the last W items (window)

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Heavy Hitters Problem: Find all items i such that $f_i > m/k$ for some fixed k.

Heavy hitters are **very** frequent items.

Finding Majority Element

Majority element problem:

- Offline: given an array/list A of m integers, is there an element that occurs more than $m/2$ times in A?
- Streaming: is there an i such that $f_i > m/2$?

Finding Majority Element

Streaming-Majority:

```
    c = 0,  s ← null
    While (stream is not empty) do
        If (e_j = s) do
            c ← c + 1
        ElseIf (c = 0)
            c = 1
            s = e_j
        Else
            c ← c − 1
    endWhile
    Output s, c
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Claim: If there is a majority element i then algorithm outputs $s = i$ and $c \geq f_i - m/2$.

Caveat: Algorithm may output incorrect element if no majority element. Can verify correctness in a second pass.

Misra-Gries Algorithm

Heavy Hitters Problem: Find all items i such that $f_i > m/k$.

```
MisraGreis(k):
    D is an empty associative array
    While (stream is not empty) do
        e_j is current item
        If (e_j is in keys(D))
            D[e_j] ← D[e_j] + 1
        Else if (|keys(A)| < k − 1) then
        D[e_j] ← 1
        Else
            for each ℓ ∈ keys(D) do
                D[ℓ] ← D[ℓ] − 1
        Remove elements from D whose counter values are 0
endWhile
For each i ∈ keys(D) set f̂_i = D[i]
For each i ∉ keys(D) set f̂_i = 0
```

Analysis

Space usage $O(k)$.

Theorem

For each $i \in [n]$: $f_i - \frac{m}{k+1} \leq \hat{f}_i \leq f_i$.

Corollary

Any item with $f_i > m/k$ is in D at the end of the algorithm.

A second pass to verify can be used to verify correctness of elements in D.

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Theorem

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Alternative view of algorithm:

- Maintains counts $C[i]$ for each i (initialized to 0). Only k are non-zero at any time.
- When new element e_j comes
  - If $C[e_j] > 0$ then increment $C[e_j]$
  - ElseIf less then k positive counters then set $C[e_j] = 1$
  - Else decrement all positive counters (exactly k of them)
- Output $\hat{f}_i = C[i]$ for each i

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- Hence total number of times α increases is at most ℓ.

Deterministic to Randomized Sketches

Cannot improve $O(k)$ space if one wants additive error of at most m/k. Nice to have a deterministic algorithm that is near-optimal

Why look for randomized solution?

- Obtain a sketch that allows for deletions
- Additional applications of sketch based solutions
- Will see **Count-Min** and **Count** sketches

Basic Hashing/Sampling Idea

Heavy Hitters Problem: Find all items i such that $f_i > m/k$.

- Let $b_1, b_2, \ldots, b_k$ be the k heavy hitters
- Suppose we pick $h : [n] \rightarrow [ck]$ for some $c > 1$
- h spreads $b_1, \ldots, b_k$ among the buckets (k balls into ck bins)
- In ideal situation each bucket can be used to count a separate heavy hitter