Frequency moments and Counting Distinct Elements

Lecture 05

September 6, 2022

Part I

Frequency Moments

Streaming model

- The input consists of m objects/items/tokens $e_1, e_2, \ldots, e_m$ that are seen one by one by the algorithm.
- The algorithm has "limited" memory say for B tokens where $B < m$ (often $B \ll m$) and hence cannot store all the input
- Want to compute interesting functions over input

Examples:

- Each token in a number from $[n]$
- High-speed network switch: tokens are packets with source, destination IP addresses and message contents.
- Each token is an edge in graph (graph streams)
- Each token in a point in some feature space
- Each token is a row/column of a matrix

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Example: $n = 5$ and stream is $4, 2, 4, 1, 1, 1, 4, 5$

Frequency Moments

- Stream consists of $e_1, e_2, \ldots, e_m$ where each e_i is an integer in $[n]$. We know n in advance (or an upper bound)
- Given a stream let f_i denote the frequency of i or number of times i is seen in the stream
- Consider vector $\mathrm{f} = (f_1, f_2, \ldots, f_n)$
- For $k \geq 0$ the k'th frequency moment $F_k = \sum_i f_i^k$. We can also consider the ℓ_k norm of f which is $(F_k)^{1/k}$.

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- $2 < k < \infty$

Frequency Moments: Questions

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Questions easy if we have memory $\Omega(n)$: store f explicitly.
Interesting when memory is $\ll n$. Ideally want to do it with $\log^c n$ memory for some fixed $c \geq 1$ ($\text{polylog}(n)$). Note that $\log n$ is roughly the memory required to store one token/number.

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Relative approximation

Let $g(\sigma)$ be a real-valued *non-negative* function over streams σ.

Definition

Let $\mathcal{A}(\sigma)$ be the real-valued output of a randomized streaming algorithm on stream σ. We say that $\mathcal{A}$ provides an (α, β) relative approximation for a real-valued function g if for all σ:

$$\Pr\left[\left|\frac{\mathcal{A}(\sigma)}{g(\sigma)} - 1\right| > \alpha\right] \leq \beta.$$

Our ideal goal is to obtain a (ϵ, δ)-approximation for any given $\epsilon, \delta \in (0, 1)$.

Additive approximation

Let $g(\sigma)$ be a real-valued function over streams σ. If $g(\sigma)$ can be negative, focus on additive approximation.

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Let $\mathcal{A}(\sigma)$ be the real-valued output of a randomized streaming algorithm on stream σ. We say that $\mathcal{A}$ provides an (α, β) additive approximation for a real-valued function g if for all σ:

$$\Pr\left[|\mathcal{A}(\sigma) - g(\sigma)| > \alpha\right] \leq \beta.$$

When working with additive approximations some normalization/scaling is typically necessary. Our ideal goal is to obtain a (ϵ, δ)-approximation for any given $\epsilon, \delta \in (0, 1)$.

Part II

Estimating Distinct Elements

Distinct Elements

Given a stream σ how many distinct elements did we see?

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Offline solution? via Dictionary data structure

Offline Solution

DistinctElements

```
Initialize dictionary D to be empty
k ← 0
While (stream is not empty) do
    Let e be next item in stream
    If (e ∉ D) then
        Insert e into D
        k ← k + 1
EndWhile
Output k
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Which dictionary data structure?

- Binary search trees: space $O(k)$ and total time $O(m \log k)$
- Hashing: space $O(k)$ and expected time $O(m)$.

Hashing based idea

- Use hash function $h : [n] \rightarrow [N]$ for some N polynomial in n.
- Store only the minimum hash value seen. That is $\min_{e_i} h(e_i)$. Need only $O(\log n)$ bits since numbers are in range $[N]$.

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Question: why is this good?

- Assume *idealized* hash function: $h : [n] \to [0, 1]$ that is fully random over the real interval
- Suppose there are k distinct elements in the stream
- What is the expected value of the minimum of hash values?

Analyzing idealized hash function

Lemma

Suppose $X_1, X_2, \ldots, X_k$ *are random variables that are independent and uniformaly distributed in* $[0, 1]$ *and let* $Y = \min_i X_i$. *Then* $E[Y] = \frac{1}{(k+1)}$.

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DistinctElements

```
Assume ideal hash function h : [n] → [0,1]
y ← 1
While (stream is not empty) do
    Let e be next item in stream
    y ← min(y, h(e))
EndWhile
Output 1/y − 1
```

Analyzing idealized hash function

Lemma

Suppose $X_1, X_2, \ldots, X_k$ are random variables that are independent and uniformaly distributed in $[0, 1]$ and let $Y = \min_i X_i$. Then $E[Y] = \frac{1}{(k+1)}$. And $Var(Y) = \frac{k}{(k+1)^2(k+2)} \leq \frac{1}{(k+1)^2}$.

$\Pr[Y \leq t] = 1 - (1 - t)^k$ for $t \in [0, 1]$. Hence probability density function of Y is $k(1 - t)^{k-1}$. Thus, $E[Y] = \int_0^1 tk(1 - t)^{k-1} dt$ and $E[Y^2] = \int_0^2 t^2 k(1 - t)^{k-1} dt$. Change variable: $z = (1 - t)$ to integrate easily.

Analyzing idealized hash function

Apply standard methodology to go from exact statistical estimator to good bounds:

- average h parallel and independent estimates to reduce variance
- apply Chebyshev to show that the average estimator is a $(1+\epsilon)$-approximation with constant probability
- use preceding and median trick with $O(\log 1/\delta)$ parallel copies to obtain a $(1+\epsilon)$-approximation with probability $(1-\delta)$

Averaging and reducing variance

1. Run basic estimator independently and in parallel h times to obtain $X_1, X_2, \ldots, X_h$
2. Let $Z = \frac{1}{h} X_i$
3. Output $\frac{1}{Z} - 1$

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Choosing $h = 1/(\eta\epsilon^2)$ and using Chebyshev:
$\Pr\left[|Z - \frac{1}{k+1}| \geq \frac{\epsilon}{k+1}\right] \leq \eta$.

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Hence $\Pr\left[|(\frac{1}{Z} - 1) - k|\right] \geq O(\epsilon)k \leq \eta$.

Repeat $O(\log 1/\delta)$ times and output median. Error probability $< \delta$.

Algorithm via regular hashing

Do not have idealized hash function.

- Use $h : [n] \to [N]$ for appropriate choice of N
- Use pairwise independent hash family $\mathcal{H}$ so that random $h \in \mathcal{H}$ can be stored in small space and computation can be done in small memory and fast

Several variants of idea with different trade offs between

- memory
- time to process each new element of the stream
- approximation quality and probability of success