Locality Sensitive Hashing

Lecture 14

October 13, 2020

Near-Neighbor Search

Collection of n points $\mathcal{P} = \{x_1, \ldots, x_n\}$ in a metric space.

NNS: preprocess $\mathcal{P}$ to answer near-neighbor queries: given query point y output $\arg\min_{x \in \mathcal{P}} \text{dist}(x, y)$

c**-approximate NNS:** given query y, output x such that $\text{dist}(x, y) \leq c \min_{z \in \mathcal{P}} \text{dist}(z, y)$. Here $c > 1$.

$c = \underline{\underline{1+\epsilon}}$

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Beating brute force is hard if one wants near-linear space!

NNS in Euclidean Spaces

Collection of n points $\mathcal{P} = \{x_1, \ldots, x_n\}$ in $\mathbb{R}^d$.
$\text{dist}(x, y) = \|x - y\|_2$ is Euclidean distance

- $d = 1$. Sort and do binary search. $O(n)$ space, $O(\log n)$ query time.
- $d = 2$. Voronoi diagram. $O(n)$ space $O(\log n)$ query time.

  

  (Figure from Wikipedia)
- Higher dimensions: Voronoi diagram size grows as $n^{\lfloor d/2 \rfloor}$.

NNS in Euclidean Spaces

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Assume n and d are large.

- Linear search with no data structures: $\Theta(nd)$ time, storage is $\Theta(nd)$
- Exact NNS: either query time or space or both are exponential in dimension d
- $(1 + \epsilon)$-approximate NNS for dimensionality reduction: reduce d to $O(\frac{1}{\epsilon^2} \log n)$ using JL but exponential in d is still impractical
- Even for approximate NNS, beating nd query time while keeping storage close to $O(nd)$ is non-trivial!

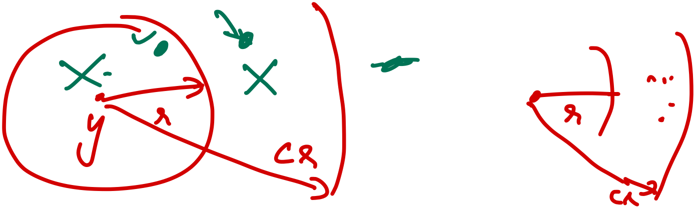
Approximate NNS

Focus on c-approximate NNS for some small $c > 1$

Simplified problem: given query point y and fixed radius $r > 0$, distinguish between the following two scenarios:

- if there is a point $x \in \mathcal{P}$ such $\text{dist}(x, y) \leq r$ output a point x' such that $\text{dist}(x', y) \leq cr$
- if $\text{dist}(x, y) \geq cr$ for all $x \in \mathcal{P}$ then recognize this and fail

Algorithm allowed to make a mistake in intermediate case



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Algorithm allowed to make a mistake in intermediate case

Can use binary search and above procedure to obtain c-approximate NNS.

Part I

LSH Framework

LSH Approach for Approximate NNS

[Indyk-Motwani'98]

Initially developed for NNSearch in high-dimensional Euclidean space and then generalized to other similarity/distance measures.

Use **locality-sensitive hashing** to solve simplified decision problem

Definition

A family of hash functions is (r, cr, p_1, p_2)-LSH with $p_1 > p_2$ and $c > 1$ if h drawn randomly from the family satisfies the following:

- $\Pr[h(x) = h(y)] \geq p_1$ when $\text{dist}(x, y) \leq r$ $p_1 = 0.3$ 1
- $\Pr[h(x) = h(y)] \leq p_2$ when $\text{dist}(x, y) \geq cr$ $p_2 = 0.1$ 0

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Key parameter: the gap between p_1 and p_2 measured as $\rho = \frac{\log p_1}{\log p_2}$

LSH Example: Hamming Distance

n points $x_1, x_2, \ldots, x_n \in \{0,1\}^d$ for some large d

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Question: What is a good (r, cr, p_1, p_2)-LSH? What is ρ?

Pick a random coordinate: Hash family $= \{h_i \mid i = 1, \ldots, d\}$ where $h_i(x) = x_i$

- Suppose $\text{dist}(x, y) \leq r$ then $\Pr[h(x) = h(y)] \geq (d - r)/d \geq 1 - r/d \simeq e^{-r/d}$
- Suppose $\text{dist}(x, y) \geq cr$ then $\Pr[h(x) = h(y)] \leq 1 - cr/d \simeq e^{-cr/d}$

LSH Example: Hamming Distance

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011011001

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2

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- Suppose $\text{dist}(x, y) \geq cr$ then $\Pr[h(x) = h(y)] \leq 1 - cr/d \simeq e^{-cr/d}$

Therefore $\rho = \frac{\log p_1}{\log p_2} \leq 1/c$

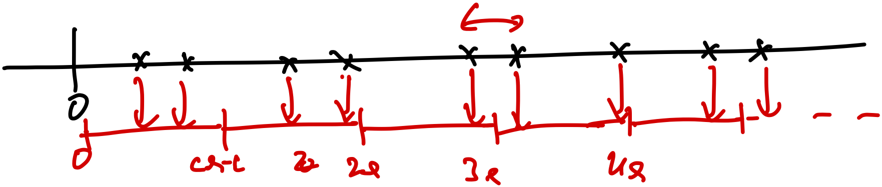
$$\frac{\ln p_1}{\ln p_2} = \frac{1}{1.1}$$

LSH Example: 1-d

1.5

n points on line and distance is Euclidean

Question: What is a good LSH? $\leq r$ $\geq cr$



w

$w \in_R [0, r]$

$p_1 \geq 1 - \frac{1}{c}$ $p_2 = 0$

LSH Example: 1-d

n points on line and distance is Euclidean

Question: What is a good LSH?

Grid line with cr units.

- No two far points will be in same bucket and hence $p_2 = 0$
- But close by points may be in different buckets. So do a random shift of grid to ensure that $p_1 \geq (1 - 1/c)$.

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- But close by points may be in different buckets. So do a random shift of grid to ensure that $p_1 \geq (1 - 1/c)$.

Main difficulty is in higher dimensions but above idea will play a role.

LSH Approach for Approximate NNS

Use **locality-sensitive hashing** to solve simplified decision problem

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Key parameter: the gap between p_1 and p_2 measured as $\rho = \frac{\log p_1}{\log p_2}$ usually small.

Two-level hashing scheme:

- Amplify basic locality sensitive hash family to create better family by repetition
- Use several copies of amplified hash functions

Amplification

Fix some r. Pick k independent hash functions $h_1, h_2, \ldots, h_k$. For each x set

$$g(x) = h_1(x)h_2(x)\ldots h_k(x)$$

$g(x)$ is now the larger hash function

- If $\mathrm{dist}(x, y) \le r$: $\mathbf{Pr}[g(x) = g(y)] \ge p_1^k$
- If $\mathrm{dist}(x, y) \ge cr$: $\mathbf{Pr}[g(x) = g(y)] \le p_2^k$

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$$p_2^k = \frac{1}{n}.$$

$$\Rightarrow k = \frac{-\log n}{\ln p_2}$$

$$p_1^k = \frac{1}{n^{\frac{\ln p_1}{\ln p_2}}} = \frac{1}{n^\rho}$$

Choose k such that $p_2^k \simeq 1/n$ so that expected number of far away points that collide with query y is ≤ 1. Then $p_1^k = 1/n^\rho$.

Multiple hash tables

- If $\text{dist}(x, y) \leq r$: $\Pr[g(x) = g(y)] \geq p_1^k$
- If $\text{dist}(x, y) \geq cr$: $\Pr[g(x) = g(y)] \leq p_2^k$

Choose k such that $p_2^k \simeq 1/n$ so that expected number of far away points that collide with query y is ≤ 1. Then $p_1^k = 1/n^\rho$.
$k = \frac{\log n}{\log(1/p_2)}$. Then $p_1^k = 1/n^\rho$ which is also small.
To make good point collide with y choose $\underline{\underline{L \simeq n^\rho}}$ hash functions $g_1, g_2, \ldots, g_L$

- $L \simeq n^\rho$ hash tables
- Storage: $nL = n^{1+\rho}$ (ignoring log factors)
- Query time: $kL = kn^\rho$ (ignoring log factors)

$k = \frac{-\log n}{\ln p_2}$

Details

What is the range of each g_i? A k tuple $(h_1(x), h_2(x), \ldots, h_k(x))$. Hence depends on range of the h's.

$h : \mathcal{U} \rightarrow [m]$

$g : \quad [m]^k$

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What is the range of each g_i? A k tuple $(h_1(x), h_2(x), \ldots, h_k(x))$. Hence depends on range of the h's.

We leave the range implicit. Say range of g_i is $[m^k]$ where range of each h is $[m]$. We only store non-empty buckets of each g_i and there can be at most n of them. For each g_i can use another hash function ℓ_i that maps m^k to $[n]$.

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So what is actually stored?

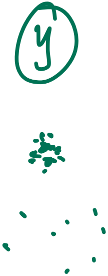
- L hash tables one for each g_i using chaining
- Each item x in database is hashed and stored in each of the L tables.
- Total storage $O(Ln)$
- Time to hash an item: Lk evaluations of basic LSH functions h_j

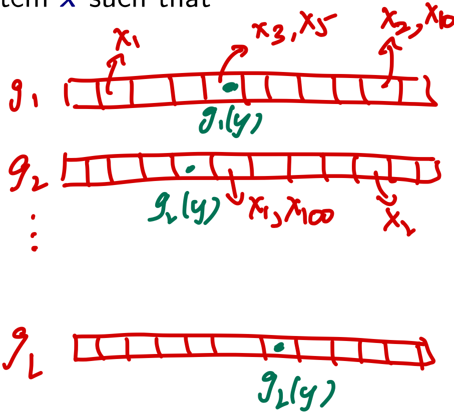
Query

$x_1, x_2, \ldots, x_n$

Given new point y how to query?

- Hash y using g_i for $1 \leq i \leq L$
- For each i check all items in bucket of $g_i(y)$ and compute all their distances and output first item x such that $\mathrm{dist}(x, y) \leq cr$.
- If no item found report FAIL





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- Hash y using g_i for $1 \leq i \leq L$
- For each i check all items in bucket of $g_i(y)$ and compute all their distances and output first item x such that $\text{dist}(x, y) \leq cr$.
- If no item found report FAIL

What if too many items collide with y? How do we bound query time?

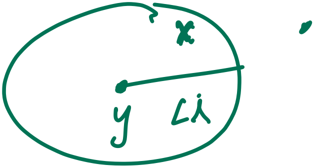
Fix: Stop search after comparing with $\Theta(L)$ items and report failure

Analysis

Query correctly fails if no item x such that $\mathrm{dist}(x, y) \leq cr$

If query outputs a point x then $\mathrm{dist}(x, y) \leq cr$

Main issue: What is the probability that there be a good point x^* such that $\mathrm{dist}(x^*, y) \leq r$ and algorithm fails?



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Two reasons

- x^* does not collide with y
- too many bad points (more than $\mathbf{10L}$ collide with y and cause query algorithm to stop and fail without discovering x^*)

X

2

Analysis

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Two reasons

- x^* does not collide with y
- too many bad points (more than $10L$ collide with y and cause query algorithm to stop and fail without discovering x^*)

First issue:

$L = n^\rho$

$$\Pr[g_i(x^*) = g_i(y)] = p_1^k \geq 1/n^\rho$$

If $L > 10n^\rho$ then $\Pr[g_i(x^*) \neq g_i(y) \forall i] \leq 1/10$.

Analysis

Main issue: What is the probability that there be a good point x^* such that $\text{dist}(x, y) \leq r$ and algorithm fails?
Two reasons

- x^* does not collide with y
- too many bad points (more than $10L$ collide with y and cause query algorithm to stop and fail without discovering x^*)

Second issue: let x be a bad point, that is $\text{dist}(x, y) > cr$

$\Pr[g_i(x) = g_i(y)] = p_2^k \leq 1/n$ by choice of k

Hence expected number of bad points that collide with y in any table is ≤ 1. Hence expected number of bad points that collide with y in all tables is at most L. By Markov, probability of more than $10L$ colliding with y is at most $1/10$

Analysis

Hence query for y succeeds with probability $1 - 2/10 \geq 4/5$.

Query time:

- Hashing y in L tables with $g_1, g_2, \ldots, g_L$ where each g_i is a k tuple of basic LSH functions. Hence $kL = kn^{\rho}$.
- Compute $d(y, x)$ for at most $O(L)$ points so total of $O(L)$ distance computations.

Amplify success probability to $1 - (1/5)^t$ by constructing t copies

Data structure only for one radius r. Need separate data structure for geometrically increasing values of r in some range $[r_{\min}, r_{\max}]$

Part II

LSH for Hamming Cube

Hamming Distance

n points $x_1, x_2, \ldots, x_n \in \{0,1\}^d$ for some large d

$\underline{2}^d \geq \underline{n}$

$d \gg \log^c n$

$\text{dist}(x, y)$ is the number of coordinates in which x, y differ

Recall that minhash and simhash reduce to Hamming distance estimation

Closely related to more general ℓ_1 distance (ideas carry over)

Question: What is a good (r, cr, p_1, p_2)-LSH? What is ρ?

$$\|x-y\|_1 = \sum_{i=1}^{d} |x_i - y_i|$$

LSH for Hamming Cube

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Pick a random coordinate. Hash family $= \{h_i \mid i = 1, \ldots, d\}$ where $h_i(x) = x_i$

$d=12$ 100101100001

$\mathcal{H} = \{h_1, h_2, \ldots, h_d\}$.

$h_i(x) = x_i$

$h_i(x) = h_i(y)$ iff $x_i = y_i$

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Suppose $\text{dist}(x, y) \leq r$ then

$$\Pr[h(x) = h(y)] \geq (d - r)/d \geq \underline{\underline{1 - r/d}} \simeq e^{-r/d} \approx p_1$$

$$1 - \frac{r}{d} + \frac{r^2}{d^2} \cdots$$

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$$\Pr[h(x) = h(y)] \geq (d - r)/d \geq 1 - r/d \simeq e^{-r/d}$$

$$\geq e^{-r/d} - \frac{r^2}{d^2}$$

Suppose $\text{dist}(x, y) \geq cr$ then

$$\Pr[h(x) = h(y)] \leq 1 - cr/d \leq e^{-cr/d}$$

Therefore $\rho = \frac{\log p_1}{\log p_2} \leq 1/c$

n^ρ

LSH for Hamming Cube

dn. space
dn query

$\rho = 1/c$

Say $c = 2$ meaning we are setting for a 2-approximate near neighbor

- query time is $\tilde{O}(d\sqrt{n})$
- space is $\tilde{O}(dn + n\sqrt{n})$

while exact/brute force requires $O(nd)$ and $O(nd)$. Thus improved query time at expense of increased space.

$\rho = \frac{1}{2}$

Space $= nL = n^{1+\rho}$

Query $= kL \simeq kn^{\rho} \cdot$ distance

LSH for Hamming Cube

$\rho = 1/c$

$\rho = \frac{1}{1.1}$ $c = 1.1$

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- query time is $\tilde{O}(d\sqrt{n})$ $n^\rho = n^{\frac{1}{1.1}}$
- space is $\tilde{O}(dn + n\sqrt{n})$ $n n^{\frac{1}{1.1}}$

while exact/brute force requires $O(nd)$ and $O(nd)$. Thus improved query time at expense of increased space.

Questions:

- Is c-approximation good in "high"-dimensions?
- Isn't space a big bottleneck?

Practice: use heuristic choices to settle for reasonable performance. LSH allows for a high-level non-trivial tradeoff between approximation and query time which is not apriori obvious

Part III

LSH for Euclidean Distances

LSH for Euclidean Distances

Now $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$ and $\text{dist}(x, y) = \|x - y\|_2$

First do dimensionality reduction (JL) to reduce d (if necessary) to $O(\log n)$ (since we are using c-approximation anyway)

LSH for Euclidean Distances

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What is a good basic locality-sensitive hashing scheme? That is, we want a hashing approach that makes nearby points more likely to collide than farther away points.

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Projections onto random lines plus bucketing

LSH for Euclidean Distances

Recall we are interested in (r, cr, p_1, p_2) lsh family for a radius r

Consider hash family with two parameters $\bar{a}, w$ where a is a random unit vector (line) in $\mathbb{R}^d$ and w is a uniform number from $[0, r]$

$$h_{a,w}(x) = \lfloor \frac{x \cdot a + w}{r} \rfloor$$

In other words we consider r length buckets on the line defined by vector a where the origin of the bucketing is via a random shift w

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$\rho < 1/c$ for this scheme though it is quite close to $1/c$.

Can achieve $\rho = (1 + o(1))\frac{1}{c^2}$ using more advanced schemes and this is close to optimal modulo constant factors.