Similarity Estimation

Lecture 13

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- find all "similar" items (application: duplicate detection in documents)
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Comparing two items expensive. Comparing all pairs, infeasible.

High-level Ideas

- How to measure similarity/dissimilarity? Proxy functions for estimating/capturing similarity
- Focus only on *highly* similar items rather than try to find similarity for all pairs
- Compression/sketching/hashing to create compact representations of objects
- Fast/approximate near-neighbor search via ideas such as locality-sensitive-hashing, clustering etc

Topics

- Jaccard similarity for sets and minhash
- Angular distance and simhash
- Locality-sensitive hashing

Part I

Jaccard Similarity and Min-wise independent Hashing

Set Similarity

Motivation: How do we detect near-duplicate text documents? Web pages, papers, homeworks, . . .?

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Model documents as (multi)sets of "words" or more generally "shingles"

- A very large set of words/singles
- Each document is a (small) set of words/shingles
- Large number of documents and each document is sparse in space of words/shingles

Jaccard similarity of sets

Definition: given two subsets S, T of a common universe the Jaccard similarity between S and T is defined as

$$\frac{|S \cap T|}{|S \cup T|}$$

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Question: Given many documents how do we find similar documents?

Min Hashing

Let n be the size of vocabulary

For a permutation σ of $[n]$ and set S let

$$\sigma_{\min}(S) = \text{first element of } S \text{ in } \sigma$$

Formally: $\sigma_{\min}(S) = \arg\min_{i \in S} \sigma(i)$

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Example:

- $[7] = \{1, 2, 3, 5, 6, 7\}$. $\sigma = 3, 5, 7, 1, 6, 2, 4$
- $S = \{2, 3, 7\}$. $\sigma_{\min}(S) = 3$
- $T = \{2, 7\}$. $\sigma_{\min}(T) = 7$.

Min Hashing

Lemma

Let S, T be two subsets of $[n]$. Suppose σ is a random permutation of $[n]$. Then

$$\Pr[\sigma_{\min}(S) = \sigma_{\min}(T)] = \frac{|S \cap T|}{|S \cup T|}.$$

Min Hashing

- Pick ℓ random permutations $\sigma^1, \sigma^2, \ldots, \sigma^\ell$
- For each set S store a ℓ-tuple $(\sigma^1_{\min}(S), \ldots, \sigma^\ell_{\min}(S))$
- To check similarity between S and T let $s = |\{i \mid \sigma^i_{\min}(S) = \sigma^i_{\min}(T)\}|$. Output estimator $Z = \text{SIM}(S, T) = s/\ell$

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Z is an exact estimator for $\text{SIM}(S, T)$.

Exercise: Suppose $\text{SIM}(S, T) \geq \alpha$. How large should ℓ be such that $\Pr[Z < (1 - \epsilon)\alpha] < \delta$?

Min Hashing

In practice:

- Pick some sufficiently large ℓ
- Use "shingles" instead of "words": depends on application
- Store for each S the compact "sketch/signature" $(\sigma^1_{\min}(S), \ldots, \sigma^\ell_{\min}(S))$
- Do further optimizations for performance/space

See Chapter 3 in Mining Massive Data Sets book by Leskovic, Rajaraman, Ullman.

Random permutation?

Random permutation like a random hash function is complex

- Cannot store compactly
- Computing $\sigma_{\min}(S)$ expensive

Need pseudorandom permutations that suffice.

Minwise Independent Permutations

[Broder-Charikar-Frieze-Mitzemacher]

Given n, S_n is the set of $n!$ permutations

Want a family $\mathcal{F} \subseteq S_n$ of permutations such that picking a random σ from $\mathcal{F}$ behaves like a random permutation (uniformly chosen from S_n)

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Definition

A family $\mathcal{F} \subseteq S_n$ is a minwise independent family of permutations if for every $X \subseteq [n]$ and $a \in X$, for a σ chosen uniformly from $\mathcal{F}$,

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- There exist minwise independent families of size 4^n
- Any minwise independent family must have size $e^{(1-o(1))n}$

Hence we need to relax the requirement further.

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Two relaxations:

- ϵ-approximate minwise independence.

$$\frac{1-\epsilon}{|X|} \leq \Pr[\sigma_{\min}(X) = a] \leq \frac{1+\epsilon}{|X|}.$$

- Need condition to hold only for sets X where $|X| \leq k$ for some $k < n$. Sufficient for applications where sets are much smaller

Relaxation of Minwise Independence

Definition

A family $\mathcal{F} \subseteq S_n$ is (ϵ, k) min-wise independent family if for all $X \subset [n]$ such that $|X| \leq k$, if σ is chosen uniformly from $\mathcal{F}$,

$$\frac{1-\epsilon}{|X|} \leq \Pr[\sigma_{\min}(X) = a] \leq \frac{1+\epsilon}{|X|}.$$

Minwise Independence and Hashing

Question: Is there a connection between minwise independent permutations and hashing?

Suppose $\mathcal{H}$ is a family of t-wise independent hash functions from $[n]$ to $[n]$. Let $h \in \mathcal{H}$. Why is h not a permutation?

Minwise Independence and Hashing

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Suppose $\mathcal{H}$ is a family of t-wise independent hash functions from $[n]$ to $[n]$. Let $h \in \mathcal{H}$. Why is h not a permutation? Because of collisions

Suppose $h : [n] \rightarrow [m]$ where $m \gg n$ then h has very low probability of collisions. Then would h behave like a minwise independent permutation?

Minwise Independence and Hashing

Theorem (Indyk)

Let $\mathcal{H}$ be a t-wis independent family of hash functions from $[n]$ to $[n]$ where $t = \Omega(\log \frac{1}{\epsilon})$. Then $\mathcal{H}$ is a (ϵ, k) minwise-independent family of permutations for $k = \Omega(\epsilon n)$.

Thus hash functions from $[n]$ to $[n]$ effectively suffice for minwise independence and can be used in minhashing.

Minwise independence and Distinct Elements

Do you see connection between minwise independent permutations/hashing and Distinct Element sampling?

Exercise: How would you used minwise independent permutations to sample near-uniformly from the set of distinct elements in a stream?

Part II

Angular Distance and Simhash

Angular distance

Given a collection of vectors $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^d$ representing some data objects.

Two vectors u, v "similar" if they point roughly in the same direction

Define $\text{dist}(u, v) = \theta(u, v)/\pi$ where $\theta(u, v)$ is angle between vectors u and v. Assuming u, v are unit vectors wlog we have $u \cdot v = cos(\theta(u, v))$.
Similarity is $1 - \text{dist}(u, v)$

Sim Hash

[Charikar] as a special case of a connection between rounding algorithms and hashing

- Pick random hyperplane/unit vector $\boldsymbol{r}$
- For each $\boldsymbol{v_i}$ set $\boldsymbol{h_r(v_i)} = \text{sign}(\boldsymbol{r} \cdot \boldsymbol{v_i})$

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Lemma

$\mathbf{Pr}[h_r(v_i) = h_r(v_j)] = \theta(v_i, v_j)/\pi.$

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- Pick random hyperplane/unit vector r
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Lemma

$\Pr[h_r(v_i) = h_r(v_j)] = \theta(v_i, v_j)/\pi$.

Using several random hyperplanes $r_1, r_2, \ldots, r_\ell$ we create a compact hash value/sketch for angle similarity. Need need pseudorandom hyperplanes ...

A general observation

For Jaccard similarity and angular similarity we had the property that there is a family of hash functions $\mathcal{H}$ such that for h chosen randomly from $\mathcal{H}$

$$\Pr[h(A) = h(B)] = \text{sim}(A, B)$$

A general observation

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For Jaccard similarity and angular similarity we had the property that there is a family of hash functions $\mathcal{H}$ such that for h chosen randomly from $\mathcal{H}$

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Lemma (Charikar)

If there is a hash family for a similarity measure $sim(\cdot, \cdot)$ with the preceding property then $d(\cdot, \cdot) = 1 - sim(\cdot, \cdot)$ is a metric and further d is embeddable in generalized Hamming distance.

Part III

Similarity and Distance Measures

Similarity and Distance

Different objects and applications drive similarity measures

Similarity between x and y large implies they are close to being identical

Another common way is to use *distances* where small distances mean higher similarity

Some common measures

- Jaccard similarity measure of sets
- Cosine angle between vectors
- Distance measures: norm based measures $\|x - y\|_p$ say $p = 1, 2, \ldots$
- Hamming distance between vectors
- Edit distance between strings
- Distance measures between probability distributions: earth-mover distance, KL divergence/relative entropy (not symmetric),

For distance measures: dimensionality reduction like JL provides way to speed up pairwise distance computation.

Part IV

Near-Neighbor Search

Similarity estimation and search

Collection of data items/objects $\mathcal{D}$

We saw ways to compress objects to speed up similarity estimation between objects

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Still two problems remain:

- find all highly similar pairs — cannot do quadratic time even with compressed hashes
- new point x: want to know all points "similar" to x in $\mathcal{D}$. linear search is not feasible

Near-Neighbor Search

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Preprocess $\mathcal{D}$ using small space so that given query x, output all $y \in \mathcal{D}$ with high similarity to x (or small distance to x)

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Classical (exact) solution approaches from geometry: Voronoi diagrams, k-d trees, space partition/filling approaches.

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Modern/recent approaches: *approximate* NN search via locality-sensitive hashing (LSH), randomized k-d trees, etc

LSH approach

Initially developed for NN search in high-dimensional Euclidean space and then generalized to other similarity/distance measures.

High-level ideas:

- collection of n objects $p_1, p_2, \ldots, p_n$ in some space
- some distance/similarity measure d on pairs of objects
- create a hash function family $\mathcal{H}$ with the property that each hash function h has "locality" preserving property
- h maps points similar to each other (or closer in distance) to the same bucket with higher probability than it would map points that are not so similar
- Use multiple independent hash functions to create a data structure
- Hashing family depends on the similarity/distance measure