Subspace Embeddings for Regression

Lecture 12

October 1, 2020

Subspace Embedding

Question: Suppose we have linear subspace E of $\mathbb{R}^n$ of dimension d. Can we find a projection $\Pi : \mathbb{R}^d \to \mathbb{R}^k$ such that for *every* $x \in E$, $\|\Pi x\|_2 = (1 \pm \epsilon)\|x\|_2$?

- Not possible if $k < d$.
- Possible if $k = \ell$. Pick Π to be an orthonormal basis for E. **Disadvantage:** This requires knowing E and computing orthonormal basis which is slow.

What we really want: *Oblivious* subspace embedding ala JL based on random projections

Oblivious Supspace Embedding

Theorem

Suppose E is a linear subspace of $\mathbb{R}^n$ of dimension d. Let Π be a DJL matrix $\Pi \in \mathbb{R}^{k \times d}$ with $k = O(\frac{d}{\epsilon^2} \log(1/\delta))$ rows. Then with probability $(1 - \delta)$ for every $x \in E$,

$$\|\frac{1}{\sqrt{k}} \Pi x\|_2 = (1 \pm \epsilon)\|x\|_2.$$

In other words JL Lemma extends from one dimension to arbitrary number of dimensions in a graceful way.

Part I

Faster algorithms via subspace embeddings

Linear model fitting

An important problem in data analysis

- n data points
- Each data point $\mathbf{a}_i \in \mathbb{R}^d$ and real value b_i. We think of $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,d})$. Interesting special case is when $d = 1$.
- What model should one use to explain the data?

Linear model fitting

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- What model should one use to explain the data?

Simplest model? Affine fitting. $b_i = \alpha_0 + \sum_{j=1}^{d} \alpha_j a_{i,j}$ for some real numbers $\alpha_0, \alpha_1, \ldots, \alpha_d$. Can restrict to $\alpha_0 = 0$ by lifting to $d + 1$ dimensions and hence linear model.

Linear model fitting

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But data is noisy so we won't be able to satisfy all data points even if true model is a linear model. How do we find a good linear model?

Regression

A

$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} = b$

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- Each data point $\mathbf{a}_i \in \mathbb{R}^d$ and real value b_i. We think of $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,d})$.

$\langle a_i^T, x \rangle = b_i$

Linear model fitting: Find real numbers $\alpha_1, \ldots, \alpha_d$ such that $b_i \simeq \sum_{j=1}^{d} \alpha_j a_{i,j}$ for all points.

$Ax = b$

Let A be matrix with one row per data point a_i. We write $x_1, x_2, \ldots, x_d$ as variables for finding $\alpha_1, \ldots, \alpha_d$.

Ideally: Find $x \in \mathbb{R}^d$ such that $Ax = b$

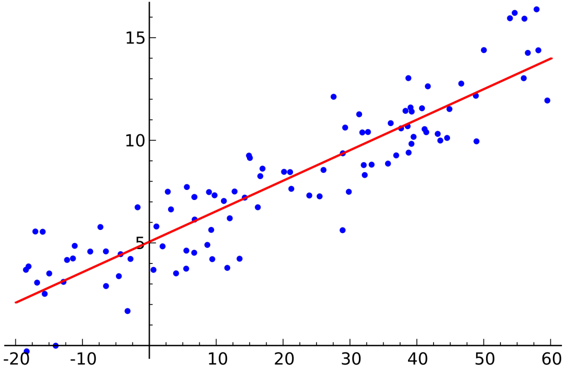
Best fit: Find $x \in \mathbb{R}^d$ to minimize $Ax - b$ under some norm.

- $\|Ax - b\|_\infty$, $\|Ax - b\|_2$, $\|Ax - b\|_1$

Linear least squares/Regression

Linear least squares: Given $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^d$ find x to minimize $\|Ax - b\|_2$. Optimal estimator for certain noise models

Interesting when $n \gg d$ the over constrained case when there is no solution to $Ax = b$ and want to find best fit.



Wikipedia

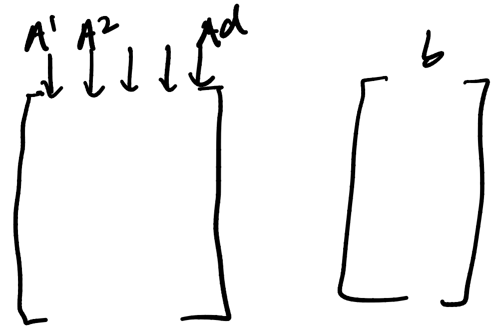
Linear least squares/Regression

Linear least squares: Given $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^{d}$ find x to minimize $\|Ax - b\|_2$.

Interesting when $n \gg d$ the over constrained case when there is no solution to $Ax = b$ and want to find best fit.

Geometrically Ax is a linear combination of columns of A. Hence we are asking what is the vector z in the column space of A that is closest to vector b in ℓ_2 norm.

Closest vector to b is the projection of b into the column space of A so it is "obvious" geometrically. How do we find it?



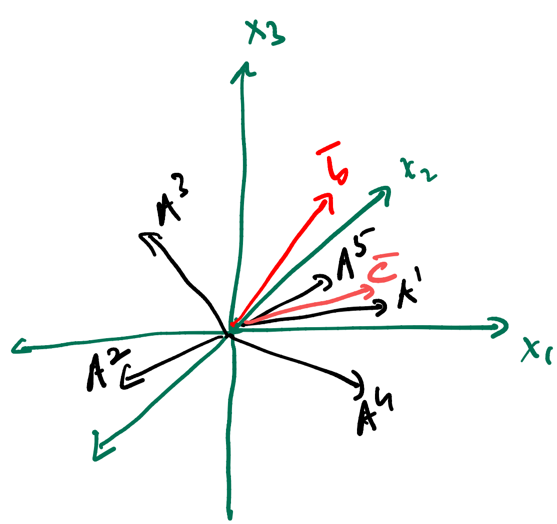
$A^1, A^2, \ldots, A^d, b \in \mathbb{R}^n.$

Fix x $\quad Ax = x_1 A^1 + x_2 A^2 + \cdots + x_d A^d$

$\|Ax - b\|_2 = 0$??

$\Longleftrightarrow$ b is in the column span of A.

Suppose b is not in the column space.
What is the ans ?



$\|Ax - b\|_2^2$

find $\bar{x}$ s.t. $\boxed{A\bar{x} = c}$

$\|b\|^2 = \|c\|_2^2 + \|b'\|^2$

Linear least squares/Regression

Linear least squares: Given $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^d$ find x to minimize $\|Ax - b\|_2$.

Geometrically Ax is a linear combination of columns of A. Hence we are asking what is the vector z in the column space of A that is closest to vector b in ℓ_2 norm.

Closest vector to b is the projection of b into the column space of A so it is "obvious" geometrically. How do we find it?

- Find an orthonormal basis $z_1, z_2, \ldots, z_r$ for the columns of A.
- Compute projection c of b to column space of A as $c = \sum_{j=1}^{r} \langle b, z_j \rangle z_j$ and output answer as $\|b - c\|_2$.
- What is x?

x is obtained by expressing c $Ax = c$

Linear least squares/Regression

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- Find an orthonormal basis $z_1, z_2, \ldots, z_r$ for the columns of A.
- Compute projection c of b to column space of A as $c = \sum_{j=1}^{r} \langle b, z_j \rangle z_j$ and output answer as $\|b - c\|_2$.
- What is x? We know that $Ax = c$. Solve linear system. Can combine both steps via SVD and other methods.

Linear least square: Optimization perspective

Linear least squares: Given $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^d$ find x to minimize $\|Ax - b\|_2$.

Optimization: Find $x \in \mathbb{R}^d$ to minimize $\|Ax - b\|_2^2$

$$\|Ax - b\|_2^2 = x^T \underbrace{A^T A}_{Q} x - 2b^T Ax + b^t b$$

The quadratic function $f(x) = x^T A^T Ax - 2b^T Ax + b^t b$ is a convex function since the matrix $A^T A$ is positive semi-definite.
$\nabla f(x) = 2A^T Ax - 2b^T A$ and hence optimum solution x^* is given by $x^* = (A^T A)^{-1} b^T A$.

Computational perspective

n large (number of data points), d smaller so A is tall and skinny.

Exact solution requires SVD or other methods. Worst case time nd^2.

Can we speed up computation with some potential approximation?

$\frac{d}{\varepsilon^2}$

nd^2

$\approx \frac{d^3}{\varepsilon^2} + nd$

Linear least squares via Subspace embeddings

Let $A^{(1)}, A^{(2)}, \ldots, A^{(d)}$ be the columns of A and let E be the subspace spanned by $\{A^{(1)}, A^{(2)}, \ldots, A^{(d)}, b\}$
Note columns are in $\mathbb{R}^n$ corresponding to n data points

$\Pi \in R^{k \times n}$

E has dimension at most $d + 1$.

Use subspace embedding on E. Applying JL matrix Π with $k = O(\frac{d}{\epsilon^2})$ rows we reduce $\{A^{(1)}, A^{(2)}, \ldots, A^{(d)}, b\}$ to $\{A'^{(1)}, A'^{(2)}, \ldots, A'^{(d)}, b'\}$ which are vectors in $\mathbb{R}^k$.

$k = \frac{c(d+1)}{\epsilon^2} \ln \frac{1}{\delta}$

Solve $\min_{x' \in \mathbb{R}^d} \|A'x' - b'\|_2$

$$\begin{array}{cccc} A^{(1)} & \cdots & A^{(d)} & b \end{array} \quad d+1$$

$\Pi \in \mathbb{R}^{k \times n}$ $\qquad k = O\left(\frac{d}{\varepsilon^2} \ln \frac{1}{\delta}\right).$

$\Pi A = A'$ $\qquad \Pi b = b'$

$$\begin{array}{cccc} A'^{(1)} & \cdots & A'^{(d)} & b' \end{array}$$

$k \times n$

$k = \frac{d}{\varepsilon^2}$

Analysis

Lemma

With probability $(1-\delta)$,

$$(1-\epsilon)\min_{x\in\mathbb{R}^d}\|Ax-b\| \le \min_{x'\in\mathbb{R}^d}\|A'x'-b'\|_2 \le (1+\epsilon)\min_{x\in\mathbb{R}^d}\|Ax-b\|$$

Analysis

Lemma

With probability $(1-\delta)$,

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With probability $(1-\delta)$ via the subpsace embedding guarantee, for all $z \in E$,

$$(1-\epsilon)\|z\|_2 \leq \|\Pi z\|_2 \leq (1+\epsilon)\|z\|_2$$

Now prove two inequalities in lemma separately using above.

Analysis

Suppose x^* is an optimum solution to $\min_x \|Ax - b\|_2$.

Let $z = Ax^* - b$. We have $\|\Pi z\|_2 \leq (1 + \epsilon)\|z\|_2$ since $z \in E$.

Analysis

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Let $z = Ax^* - b$. We have $\|\Pi z\|_2 \leq (1+\epsilon)\|z\|_2$ since $z \in E$.

Since x^* is a feasible solution to $\min_{x'} \|A'x' - b'\|$,

$$\min_{x'} \|A'x' - b'\|_2 \leq \|A'x^* - b'\|_2 = \|\Pi(Ax^* - b)\|_2 \leq (1+\epsilon)\|Ax^* - b\|_2$$

$$\|\Pi z\| \leq (1+\epsilon)\|z\|$$
$$= (1+\epsilon)\|Ax^* - b\|_2$$

Analysis

For *any* $y \in \mathbb{R}^d$, $\|\Pi A y - \Pi b\|_2 \geq (1-\epsilon)\|Ay - b\|_2$ because $Ay - b$ is a vector in E and Π preserves all of them.

z

$$\|\Pi(Ay-b)\|_2 = \|\Pi Ay - \Pi b\|_2$$

$$\|\Pi z\| \geq (1-\varepsilon)\|z\|$$

Analysis

For *any* $y \in \mathbb{R}^d$, $\|\Pi A y - \Pi b\|_2 \geq (1-\epsilon)\|Ay - b\|_2$ because $Ay - b$ is a vector in E and Π preserves all of them.

Let y^* be optimum solution to $\boxed{\min_{x'} \|A'x' - b'\|_2.}$ Then
$\|\Pi(Ay^* - b)\|_2 \geq (1-\epsilon)\|Ay^* - b\|_2 \geq (1-\epsilon)\|Ax^* - b\|_2$

Running time

Reduce problem for d vectors in $\mathbb{R}^n$ to d vectors in $\mathbb{R}^k$ where $k = O(d/\epsilon^2)$.

Computing $\Pi A, \Pi b$ can be done in $\mathrm{nnz}(A)$ via sparse/fast JL (input sparsity time).

Need to solve least squares on A', b' which can be done in $O(d^3/\epsilon^2)$ time.

Essentially reduce n to d/ϵ^2. Useful when $n \gg d/\epsilon^2$ (for this ϵ should not be too small)

$\underline{\underline{\Pi A}}$

Further improvement

Reduced dimension of vectors from $\mathbb{R}^n$ to $\mathbb{R}^k$ where $k = O(d/\epsilon^2)$.

For small ϵ a dependence of $1/\epsilon^2$ is not so good. Can we improve?

Can use Π with $k = O(d/\epsilon)$.

- Suffices if Π has $1/10$-approximate subspace embedding property *and* property of preserving matrix multiplication
- $(\Pi A)^T(\Pi A)$ has small condition number
- Use Π that has $1/10$-approximate subspace embedding property and then use gradient descent whose convergence depends on condition number of A.

Other uses of JL/subspace embeddings in numerical linear algebra

- Approximate matrix multiplication
- Low rank approximation and SVD
- Compressed Sensing