Applications of CountMin and Count Sketches

Lecture 10

September 24, 2020

CountMin Sketch

```
CountMin-Sketch(w, d):
    h_1, h_2, ..., h_d are pair-wise independent hash functions
        from [n] → [w].
    While (stream is not empty) do
        e_t = (i_t, Δ_t) is current item
        for ℓ = 1 to d do
            C[ℓ, h_ℓ(i_j)] ← C[ℓ, h_ℓ(i_j)] + Δ_t
    endWhile
    For i ∈ [n] set x̃_i = min_{ℓ=1}^{d} C[ℓ, h_ℓ(i)].
```

Counter $C[\ell, j]$ simply counts the sum of all x_i such that $h_\ell(i) = j$. That is,

$$C[\ell, j] = \sum_{i : h_\ell(i) = j} x_i.$$

Summarizing

Lemma

Let $d = \Omega(\log \frac{1}{\delta})$ *and* $w > \frac{2}{\epsilon}$*. Then for any fixed* $i \in [n]$, $x_i \leq \tilde{x}_i$ *and*

$$\Pr[\tilde{x}_i \geq x_i + \epsilon \|x\|_1] \leq \delta.$$

Corollary

With $d = \Omega(\ln n)$ *and* $w = 2/\epsilon$*, with probability* $(1 - \frac{1}{n})$ *for all* $i \in [n]$*:*

$$\tilde{x}_i \leq x_i + \epsilon \|x\|_1.$$

Total space: $O(\frac{1}{\epsilon} \log n)$ counters and hence $O(\frac{1}{\epsilon} \log n \log m)$ bits.

Count Sketch

Count-Sketch(w, d):

```
    h_1, h_2, ..., h_d are pair-wise independent hash functions
        from [n] → [w].
    g_1, g_2, ..., g_d are pair-wise independent hash functions
        from [n] → {−1, 1}.
    While (stream is not empty) do
        e_t = (i_t, Δ_t) is current item
        for ℓ = 1 to d do
            C[ℓ, h_ℓ(i_j)] ← C[ℓ, h_ℓ(i_j)] + g(i_t)Δ_t
    endWhile
    For i ∈ [n]
        set x̃_i = median{g_1(i)C[1, h_1(i)], ..., g_ℓ(i)C[ℓ, h_ℓ(i)]}.
```

Summarizing

Lemma

Let $d \geq 4 \log \frac{1}{\delta}$ *and* $w > \frac{3}{\epsilon^2}$*. Then for any fixed* $i \in [n]$*,* $\mathbf{E}[\tilde{x}_i] = x_i$ *and* $\Pr[|\tilde{x}_i - x_i| \geq \epsilon \|x\|_2] \leq \delta$.

Corollary

With $d = \Omega(\ln n)$ *and* $w = 3/\epsilon^2$*, with probability* $(1 - \frac{1}{n})$ *for all* $i \in [n]$*:*

$$|\tilde{x}_i - x_i| \leq \epsilon \|x\|_2.$$

Total space $O(\frac{1}{\epsilon^2} \log n)$ counters and hence $O(\frac{1}{\epsilon^2} \log n \log m)$ bits.

Part I

Applications

Heavy Hitters: Point queries

Heavy Hitters Problem: Find all items i such that $x_i > \alpha \|x\|_1$ for some fixed $\alpha \in (0, 1]$.

Approximate version: output any i such that $x_i \geq (\alpha - \epsilon)\|x\|_1$

The sketches give us a data structure such that for any $i \in [n]$ we get an estimate $\tilde{x}_i$ of x_i with additive error.

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Additional data structures to speed up above computation and reduce time/space to be proportional to $O(\frac{1}{\alpha}\text{polylog}(n))$. More tricky for Count Sketch. See notes and references

Range Queries

Range query: given $i, j \in [n]$ want to know $\sum_{i \leq \ell \leq j} x[i,j]$

Examples:

- $[n]$ corresponds to IP address space in network routing and $[i,j]$ corresponds to addresses in a range
- $[n]$ corresponds to some numerical attribute in a database and we want to know number of records within a range
- $[n]$ corresponds to the discretization of a signal value

x

f $(x_1, x_2, \ldots, x_n)$

$i < j$ 10 250

$\binom{n}{2}$

$\sum_{i \leq \ell \leq j} x_\ell$

$\tilde{x}_i = x_i \pm$ error

Sketch is small.

$\tilde{x}_{i,j} = x_{i,j} +$ additive error

$(0, 1, 10, 5, 3, 2, 1, 10, \ldots,)$

— i

— j

Range Queries

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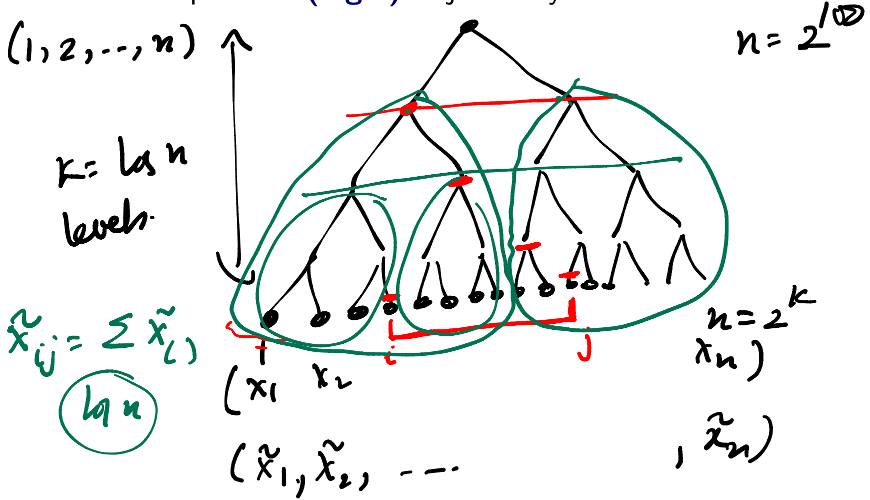
Want to create a sketch data structure that can answer range queries for any given range that is chosen *after* the sketch is done. $\Omega(n^2)$ potential queries

Range Queries

Simple idea: imagine a binary tree over $[n]$ and any interval $[i, j]$ can be broken up into $O(\log n)$ disjoint "dyadic" intervals



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Create one sketch data structure per level of binary tree

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Output estimate $\tilde{x}[i, j]$ by adding estimates for $O(\log n)$ dyadic intervals that $[i, j]$ decomposes into

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Create one sketch data structure per level of binary tree

Output estimate $\tilde{x}[i, j]$ by adding estimates for $O(\log n)$ dyadic intervals that $[i, j]$ decomposes into

To manage error choose $\epsilon' = \epsilon / \log n$: total space is $O(\alpha \log n/\epsilon)$ where α is the space for single level sketch

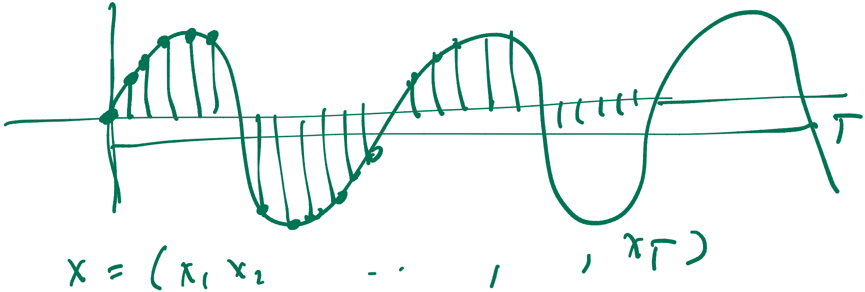
Part II

Sparse Recovery

Sparse Recovery

Sparsity is an important theme in optimization/algorithms/modeling

- Data is often *explicitly* sparse. Examples: graphs, matrices, vectors, documents (as word vectors)
- Data is often *implicitly* sparse — in a different representation the data is explicitly sparse. Examples: signals/images, topics, etc



Sparse Recovery

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Algorithmic goals

- Take advantage of sparsity to improve performance (speed, quality, memory etc)
- Find implicit sparse representation to reveal information about data. Excample: topics in documents, frequencies in Fourier analysis

Sparse Recovery

Problem: Given vector/signal $x \in \mathbb{R}^n$ find a sparse vector z such that z approximates x

More concretely: given x and integer $k \geq 1$, find z such that z has at most k non-zeroes ($\|z\|_0 \leq k$) such that $\|x - z\|_p$ is minimized for some $p \geq 1$.

Optimum offline solution: z picks the largest k coordinates of x (in absolute value)

Want to do it in streaming setting: turnstile streams and $p = 2$ and want to use $\tilde{O}(k)$ space proportional to output

$k = 3$

(0, 10, 1, 9, 0.1, -0.2, 2, 0.001, 5)

(0, 10, 0, 9, 0, 0, 0, 0, 5)

$\|x - z\|_2$

Sparse Recovery under ℓ_2 norm

Formal objective function:

$$\alpha_k = \mathrm{err}_2^k(x) = \min_{z : \|z\|_0 \le k} \|x - z\|_2$$

$$(1, 1, 1+\varepsilon, 1-\varepsilon, 1, 1-\varepsilon, 0, 0, 0, \dots, 0)$$

Sparse Recovery under ℓ_2 norm

Formal objective function:

$$\mathrm{err}_2^k(x) = \min_{z:\|z\|_0 \le k} \|x - z\|_2$$

$\mathrm{err}_2^k(x)$ is interesting only when it is small compared to $\|x\|_2$

For instance when x is uniform, say $x_i = 1$ for all i then $\|x\|_2 = \sqrt{n}$ but $\mathrm{err}_2^k(x) = \sqrt{n-k}$

$\mathrm{err}_2^k(x) = 0$ iff $\|x\|_0 \le k$ and hence related to distinct element detection

$(1, 1, 1, \ldots, 1)$

$(\underbrace{1, 1, 1, 1}_{k}, 0000)$

$\|x\|_2 = \sqrt{n}$

$\|x - z\|_2 = \sqrt{n-k}.$

Sparse Recovery under ℓ_2 norm

Theorem

There is a linear sketch with size $O(\frac{k}{\epsilon^2} polylog(n))$ that returns z such that $\|z\|_0 \le k$ and with high probability $\|x - z\|_2 \le (1 + \epsilon) err_2^k(x)$.

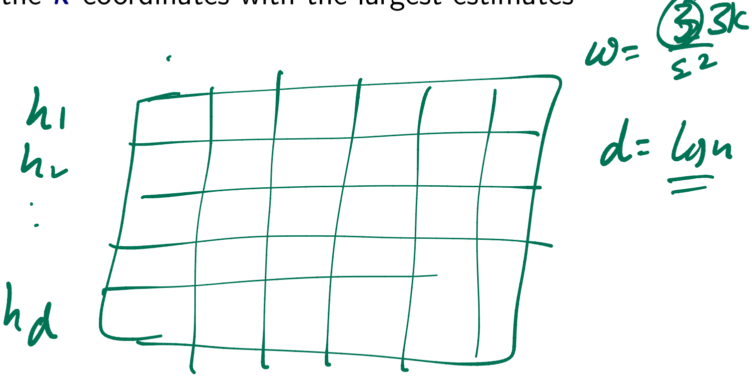
Hence space is proportional to desired output. Assumption k is typically quite small compared to n, the dimension of x.

Note that if x is k-sparse vector is exactly reconstructed

Based on CountSketch

Algorithm

- Use Count Sketch with $w = 3k/\epsilon^2$ and $d = \Omega(\log n)$.
- Count Sketch gives estimages $\tilde{x}_i$ for each $i \in n$
- Output the k coordinates with the largest estimates



Algorithm

- Use Count Sketch with $w = 3k/\epsilon^2$ and $d = \Omega(\log n)$.
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$w = \frac{3}{\epsilon^2}$

$|\tilde{x}_i - x_i| \leq \epsilon \cdot \|x\|_2$.

Intuition for analysis

- With $w = ck/\epsilon^2$ the k biggest coordinates will be spread out in their own buckets
- rest of small coordinates will be spread out evenly
- refine the analysis of Count-Sketch to carefully analyze the two scenarios

Analysis Outline

Lemma

Count-Sketch with $w = 3k/\epsilon^2$ and $d = O(\log n)$ ensures that

$$\forall i \in [n], \quad |\tilde{x}_i - x_i| \leq \frac{\epsilon}{\sqrt{k}} err_2^k(x)$$

$\varepsilon \|x\|_2$
$\varepsilon \|x-$

with high probability (at least $(1 - 1/n)$).

Lemma

Let $x, y \in \mathbb{R}^n$ such that $\|x - y\|_\infty \leq \frac{\epsilon}{\sqrt{k}} err_2^k(x)$. Then, $\|x - z\|_2 \leq (1 + 5\epsilon) err_2^k(x)$, where z is the vector obtained as follows: $z_i = y_i$ for $i \in T$ where T is the set of k largest (in absolute value) indices of y and $z_i = 0$ for $i \notin T$.

Lemmas combined prove the correctness of algorithm.

Count Sketch

Count-Sketch(w, d):
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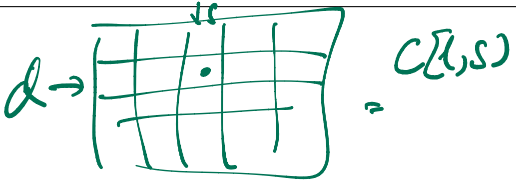
        from $[n] \to [w]$.
    $g_1, g_2, \ldots, g_d$ are pair-wise independent hash functions
        from $[n] \to \{-1, 1\}$.
    While (stream is not empty) do
        $e_t = (i_t, \Delta_t)$ is current item
        for $\ell = 1$ to d do
            $C[\ell, h_\ell(i_j)] \leftarrow C[\ell, h_\ell(i_j)] + g(i_t)\Delta_t$
    endWhile
    For $i \in [n]$
        set $\tilde{x}_i = \text{median}\{g_1(i)C[1, h_1(i)], \ldots, g_d(i)C[d, h_d(i)]\}$.



Recap of Analysis

Fix an $i \in [n]$. Let $Z_\ell = g_\ell(i)C[\ell, h_\ell(i)]$.

For $i' \in [n]$ let $Y_{i'}$ be the indicator random variable that is 1 if $h_\ell(i) = h_\ell(i')$; that is i and i' collide in h_ℓ.
$E[Y_{i'}] = E[Y_{i'}^2] = 1/w$ from pairwise independence of h_ℓ.

$$Z_\ell = g_\ell(i)C[\ell, h_\ell(i)] = g_\ell(i) \sum_{i'} g_\ell(i')x_{i'}Y_{i'}$$

Therefore,

$$E[Z_\ell] = x_i + \sum_{i' \neq i} E[g_\ell(i)g_\ell(i')Y_{i'}]x_{i'} = x_i,$$

because $E[g_\ell(i)g_\ell(i')] = 0$ for $i \neq i'$ from pairwise independence of g_ℓ and $Y_{i'}$ is independent of $g_\ell(i)$ and $g_\ell(i')$.

Recap of Analysis

$Z_\ell = g_\ell(i) C[\ell, h_\ell(i)]$. And $\mathbf{E}[Z_\ell] = x_i$.

$$
\begin{aligned}
Var(Z_\ell) &= \mathbf{E}\left[(Z_\ell - x_i)^2\right] \\
&= \mathbf{E}\left[\Big(\sum_{i' \neq i} g_\ell(i) g_\ell(i') Y_{i'} x_{i'}\Big)^2\right] \\
&= \mathbf{E}\left[\sum_{i' \neq i} x_{i'}^2 Y_{i'}^2 + \sum_{i' \neq i''} x_{i'} x_{i''} g_\ell(i') g_\ell(i'') Y_{i'} Y_{i''} x_{i'} x_{i''}\right] \\
&= \sum_{i' \neq i} x_{i'}^2 \mathbf{E}\left[Y_{i'}^2\right] \\
&\leq \|x\|_2^2 / w.
\end{aligned}
$$

Refining Analysis

$T_{\text{big}} = \{i' \mid i' \text{ is one of the } k \text{ biggest coordinates in } x\}$

$T_{\text{small}} = [n] \setminus T$

$\sum_{i' \in T_{\text{small}}} x_{i'}^2 = (\text{err}_2^k(x))^2$

α_k^2

$|\tilde{x}_i - x_i| \le \varepsilon \|x\|_2$

$\le \frac{\varepsilon}{\sqrt{k}} \alpha_k$

$(0, 1, 100, -1, 0, \ldots)$

Refining Analysis

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What is $\mathbf{Pr}\left[|Z_\ell - x_i| \geq \frac{\epsilon}{\sqrt{k}} \text{err}_2^k(x)\right]$?

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What is $\mathbf{Pr}\left[|Z_\ell - x_i| \geq \frac{\epsilon}{\sqrt{k}}\text{err}_2^k(x)\right]$?

Lemma

$\mathbf{Pr}\left[|Z_\ell - x_i| \geq \frac{\epsilon}{\sqrt{k}} err_2^k(x)\right] \leq 2/5.$

Analysis

$Z_\ell = g_\ell(i) C[\ell, h_\ell(i)]$.

$w = \frac{3k}{\epsilon^2}$

Let A_ℓ be event that $h_\ell(i') = h_\ell(i)$ for some $i' \in T_{\text{big}}, i' \neq i$

Lemma

$\Pr[A_\ell] \leq \epsilon^2/3$. *In other words with* $1 - \epsilon^2/3$ *probability no big coordinates collide with* i *under* h_ℓ.

$\frac{3k}{\epsilon^2}$

$k \cdot \frac{1}{w} = k \cdot \frac{1}{\frac{3k}{\epsilon^2}}$

$\frac{\epsilon^2}{3}$

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- $Y_{i'}$ indicator for $i' \neq i$ colliding with i. $\Pr[Y_{i'}] \leq 1/w \leq \epsilon^2/(3k)$.
- Let $Y = \sum_{i' \in T_{\text{big}}} Y_{i'}$. $\mathbf{E}[Y] \leq \epsilon^2/3$ by linearity of expectation.
- Hence $\Pr[A_\ell] = \Pr[Y \geq 1] \leq \epsilon^2/3$ by Markov

Analysis

$$Z_\ell = g_\ell(i) C[\ell, h_\ell(i)]$$
$$= x_i + \sum_{i' \in T_{\text{big}}} g_\ell(i) g_\ell(i') Y_{i'} x_{i'} + \sum_{i' \in T_{\text{small}}} g_\ell(i) g_\ell(i') Y_{i'} x_{i'}$$

Let $Z'_\ell = \sum_{i' \in T_{\text{small}}} g_\ell(i) g_\ell(i') Y_{i'}$

Lemma

$$\Pr\left[|Z'_\ell| \geq \frac{\epsilon}{\sqrt{k}} err_2^k(x)\right] \leq 1/3.$$

$E[Z'_\ell] = 0$

$\varepsilon \|x\|_2$

$\Pr[|Z'_\ell| \geq \frac{\varepsilon}{\sqrt{k}} d_k] \leq \frac{1}{3}$.

Analysis

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Lemma

$\Pr\left[|Z'_\ell| \geq \frac{\epsilon}{\sqrt{k}} err_2^k(x)\right] \leq 1/3.$

- $\mathbf{E}\left[Z'_\ell\right] = 0$
- $Var(Z'_\ell) \leq \mathbf{E}\left[(Z'_\ell)^2\right] = \sum_{i' \in T_{\text{small}}} x_{i'}^2 / w \leq \frac{\epsilon^2}{3k} (\text{err}_2^k(x))^2$
- By Cheybyshev $\Pr\left[|Z'_\ell| \geq \frac{\epsilon}{\sqrt{k}} \text{err}_2^k(x)\right] \leq 1/3.$

Analysis: Proof of lemma

Want to show:

Lemma

$$\Pr\left[|Z_\ell - x_i| \geq \frac{\epsilon}{\sqrt{k}} err_2^k(x)\right] \leq 2/5.$$

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We saw:

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$\Pr[A_\ell] \leq \epsilon^2/3$. *In other words with* $1 - \epsilon^2/3$ *probability no big coordinates collide with* i *under* h_ℓ.

Analysis: Proof of lemma

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$$= x_i + \sum_{i' \in T_{\text{big}}} g_\ell(i) g_\ell(i') Y_{i'} x_{i'} + \sum_{i' \in T_{\text{small}}} g_\ell(i) g_\ell(i') Y_{i'} x_{i'}$$

$|Z_\ell - x_i| \geq \frac{\epsilon}{\sqrt{k}} \text{err}_2^k(x)$ implies

- A_ℓ happens (that is some big coordinate collides with i in h_ℓ **or**
- $|Z'_\ell| \geq \frac{\epsilon}{\sqrt{k}} \text{err}_2^k(x)$

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Therefore, by union bound,
$\mathbf{Pr}\left[|Z_\ell - x_i| \geq \frac{\epsilon}{\sqrt{k}} \text{err}_2^k(x) \right] \leq \epsilon^2/3 + 1/3 \leq 2/5$
if ϵ is sufficiently small.

High probability estimate

Lemma

$$\mathbf{Pr}\left[|Z_\ell - x_i| \geq \frac{\epsilon}{\sqrt{k}} err_2^k(x)\right] \leq 2/5.$$

Recall $\tilde{x}_i = \text{median}\{g_1(i)C[1, h_1(i)], \ldots, g_d(i)C[d, h_d(i)]\}$.

- Hence by Chernoff bounds with $d = \Omega(\log n)$,
  $\mathbf{Pr}\left[|\tilde{x}_i - x_i| \geq \frac{\epsilon}{\sqrt{k}} \text{err}_2^k(x)\right] \leq 1/n^2$
- By union bound, with probability at least $(1 - 1/n)$,
  $|\tilde{x}_i - x_i| \leq \frac{\epsilon}{\sqrt{k}} \text{err}_2^k(x)$ for all $i \in [n]$.

La

$\forall i \quad Pr\left[|\tilde{x}_i - x_i| \geq \frac{\epsilon}{\sqrt{k}} \alpha_k\right] \leq \frac{1}{n^2}$

High probability estimate

Lemma

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Lemma

Count-Sketch with $w = 3k/\epsilon^2$ *and* $d = O(\log n)$ *ensures that* $\forall i \in [n], \quad |\tilde{x}_i - x_i| \leq \frac{\epsilon}{\sqrt{k}} err_2^k(x)$ *with high probability (at least* $(1 - 1/n)$*).*

Second lemma of outline

Lemma

Let $x, y \in \mathbb{R}^n$ such that $\|x - y\|_\infty \leq \frac{\epsilon}{\sqrt{k}} err_2^k(x)$. Then, $\|x - z\|_2 \leq (1 + 5\epsilon) err_2^k(x)$, where z is the vector obtained as follows: $z_i = y_i$ for $i \in T$ where T is the set of k largest (in absolute value) indices of y and $z_i = 0$ for $i \notin T$.

What the lemma is saying:

- $\tilde{x}$ the estimated vector of Count-Sketch approximates x very closely in *each coordinate*
- Algorithm picks the top k coordinates of $\tilde{x}$ to create z
- Then z approximates x well

$x = (100, 100, 100, 100, 0.1, 0.001, 0, 1, 0.5, \ldots)$

$k=4$

$\tilde{x} = (97, 105, 110, 83, 0.5, 0.1, 3, 0, 0 \ldots)$

$$\|\tilde{x} - x\|_\infty \le \boxed{\frac{\varepsilon}{\sqrt{k}} \alpha_k} \approx 1$$

$x = (100, 100, 100, 100, 95, 0.1, 0.001, 0, 1, 0.5, \ldots)$

$k=4$

$\tilde{x} = (97, 83, 99, 105, 90, \ldots \quad)$

95^2

$$|\tilde{x} - x_i| \le \frac{\varepsilon}{\sqrt{k}} \alpha_k$$

$100^2 + \ldots$

Second lemma of outline

Lemma

Let $x, y \in \mathbb{R}^n$ such that $\|x - y\|_\infty \leq \frac{\epsilon}{\sqrt{k}} err_2^k(x)$. Then, $\|x - z\|_2 \leq (1 + 5\epsilon) err_2^k(x)$, where z is the vector obtained as follows: $z_i = y_i$ for $i \in T$ where T is the set of k largest (in absolute value) indices of y and $z_i = 0$ for $i \notin T$.

What the lemma is saying:

- $\tilde{x}$ the estimated vector of Count-Sketch approximates x very closely in *each coordinate*
- Algorithm picks the top k coordinates of $\tilde{x}$ to create z
- Then z approximates x well

Proof is basically follows the intuition of triangle inequality

Proof of lemma

S (previously T_{big}) is set of k biggest coordinates in x
T is the set of k biggest coordinates in $y = \tilde{x}$
Let $E = \frac{1}{\sqrt{k}} \text{err}_2^k(x)$ for ease of notation.

$$(\text{err}_2^k(x))^2 = kE^2 = \sum_{i \in [n] \setminus S} x_i^2 = \sum_{i \in T \setminus S} x_i^2 + \sum_{i \in [n] \setminus (S \cup T)} x_i^2.$$

Want to bound

$$\begin{aligned} \|x - z\|_2^2 &= \sum_{i \in T} |x_i - z_i|^2 + \sum_{i \in S \setminus T} |x_i - z_i|^2 + \sum_{i \in [n] \setminus (S \cup T)} x_i^2 \\ &= \sum_{i \in T} |x_i - y_i|^2 + \sum_{i \in S \setminus T} x_i^2 + \sum_{i \in [n] \setminus (S \cup T)} x_i^2. \end{aligned}$$

Analysis continued

Want to bound

$$\begin{aligned}\|x - z\|_2^2 &= \sum_{i \in T} |x_i - z_i|^2 + \sum_{i \in S \setminus T} |x_i - z_i|^2 + \sum_{i \in [n] \setminus (S \cup T)} x_i^2 \\ &= \sum_{i \in T} |x_i - y_i|^2 + \sum_{i \in S \setminus T} x_i^2 + \sum_{i \in [n] \setminus (S \cup T)} x_i^2.\end{aligned}$$

First term: $\sum_{i \in T} |x_i - \tilde{x}_i|^2 \leq k\epsilon^2 E^2 \leq \epsilon^2 (\text{err}_2^k(x))^2$

Analysis continued

Want to bound

$$\begin{aligned}
\|x - z\|_2^2 &= \sum_{i \in T} |x_i - z_i|^2 + \sum_{i \in S \setminus T} |x_i - z_i|^2 + \sum_{i \in [n] \setminus (S \cup T)} x_i^2 \\
&= \sum_{i \in T} |x_i - y_i|^2 + \sum_{i \in S \setminus T} x_i^2 + \sum_{i \in [n] \setminus (S \cup T)} x_i^2.
\end{aligned}$$

First term: $\sum_{i \in T} |x_i - \tilde{x}_i|^2 \le k\epsilon^2 E^2 \le \epsilon^2 (\mathrm{err}_2^k(x))^2$

Third term: common to expression for $(\mathrm{err}_2^k(x))^2$

Analysis continued

Want to bound

$$\begin{aligned}
\|x - z\|_2^2 &= \sum_{i \in T} |x_i - z_i|^2 + \sum_{i \in S \setminus T} |x_i - z_i|^2 + \sum_{i \in [n] \setminus (S \cup T)} x_i^2 \\
&= \sum_{i \in T} |x_i - y_i|^2 + \sum_{i \in S \setminus T} x_i^2 + \sum_{i \in [n] \setminus (S \cup T)} x_i^2 .
\end{aligned}$$

First term: $\sum_{i \in T} |x_i - \tilde{x}_i|^2 \leq k\epsilon^2 E^2 \leq \epsilon^2 (\text{err}_2^k(x))^2$

Third term: common to expression for $(\text{err}_2^k(x))^2$

Second term: needs more care

Analysis contd

Want to bound $\sum_{i \in S \setminus T} x_i^2$

Let $\ell = |S \setminus T| \leq k$. Since $|S| = |T| = k$, $|T \setminus S| = \ell$

Coordinates in $S \setminus T$ and $T \setminus S$ must be close: within $\frac{\epsilon}{\sqrt{k}} \mathrm{err}_2^k(x)$

Analysis contd

Want to bound $\sum_{i \in S \setminus T} x_i^2$

Let $\ell = |S \setminus T| \leq k$. Since $|S| = |T| = k$, $|T \setminus S| = \ell$

Coordinates in $S \setminus T$ and $T \setminus S$ must be close: within $\frac{\epsilon}{\sqrt{k}} \mathrm{err}_2^k(x)$

Claim: Let $a = \max_{i \in S \setminus T} |x_i|$ and $b = \min_{i \in T \setminus S} |x_i|$. Then $a \leq b + 2\frac{\epsilon}{\sqrt{k}} \mathrm{err}_2^k(x)$.

Analysis contd

Want to bound $\sum_{i \in S \setminus T} x_i^2$

Let $\ell = |S \setminus T| \leq k$. Since $|S| = |T| = k$, $|T \setminus S| = \ell$

Coordinates in $S \setminus T$ and $T \setminus S$ must be close: within $\frac{\epsilon}{\sqrt{k}} \text{err}_2^k(x)$

Claim: Let $a = \max_{i \in S \setminus T} |x_i|$ and $b = \min_{i \in T \setminus S} |x_i|$. Then $a \leq b + 2\frac{\epsilon}{\sqrt{k}} \text{err}_2^k(x)$.

Therefore

$$\begin{aligned} \sum_{i \in S \setminus T} x_i^2 &\leq \ell a^2 \leq \ell (b + 2\frac{\epsilon}{\sqrt{k}} \text{err}_2^k(x))^2 \\ &\leq \ell b^2 + 4k\frac{\epsilon^2}{k} (\text{err}_2^k(x))^2 + 4kb\frac{\epsilon}{\sqrt{k}} \text{err}_2^k(x). \end{aligned}$$

Analysis contd

$$\begin{aligned}
\sum_{i \in S \setminus T} x_i^2 &\leq \ell a^2 \leq \ell(b + 2\frac{\epsilon}{\sqrt{k}}\mathrm{err}_2^k(x))^2 \\
&\leq \ell b^2 + 4k\frac{\epsilon^2}{k}(\mathrm{err}_2^k(x))^2 + 4kb\frac{\epsilon}{\sqrt{k}}\mathrm{err}_2^k(x) \\
&\leq \ell b^2 + 4\epsilon^2(\mathrm{err}_2^k(x))^2 + 4\epsilon(\sqrt{k}b)\mathrm{err}_2^k(x) \\
&\leq \ell b^2 + 8\epsilon(\mathrm{err}_2^k(x))^2 \\
&\leq \sum_{i \in T \setminus S} x_i^2 + 8\epsilon(\mathrm{err}_2^k(x))^2.
\end{aligned}$$

Exercise: Why is $\sqrt{k}b \leq \mathrm{err}_2^k(x)$? (We used it above.)

Analysis contd

$$\begin{aligned}
\|x - z\|_2^2 &= \sum_{i \in T} |x_i - z_i|^2 + \sum_{i \in S \setminus T} |x_i - z_i|^2 + \sum_{i \in [n] \setminus (S \cup T)} x_i^2 \\
&= \sum_{i \in T} |x_i - y_i|^2 + \sum_{i \in S \setminus T} x_i^2 + \sum_{i \in [n] \setminus (S \cup T)} x_i^2.
\end{aligned}$$

First term: $\sum_{i \in T} |x_i - \tilde{x}_i|^2 \leq k\epsilon^2 E^2 \leq \epsilon^2 (\text{err}_2^k(x))^2$
Third term: common to expression for $(\text{err}_2^k(x))^2$
Second term: at most $\sum_{i \in T \setminus S} x_i^2 + 8\epsilon(\text{err}_2^k(x))^2$
Hence

$$\|x - z\|_2^2 \leq (1 + 9\epsilon)(\text{err}_2^k(x))^2$$

Implies

$$\|x - z\|_2 \leq (\sqrt{1 + 9\epsilon})\text{err}_2^k(x) \leq (1 + 5\epsilon)\text{err}_2^k(x)$$

Application to signal processing

Given signal x approximate it via small number of basis signals

- Fourier analysis and Wavelets
- Useful in compression of various kinds

Application to signal processing

Given signal x approximate it via small number of basis signals

- Fourier analysis and Wavelets
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Transform x into $y = Bx$ where B is a transform and then approximate y by k-sparse vector z

To (approximately) reconstruct x, output $x' = B^{-1}z$

If Bx can be computed in streaming fashion from stream for x, we can apply preceding algorithm to obtain z

Compressed Sensing

We saw that *given* x in streaming fashion we can construct sketch that allows us to find k-sparse z that approximates x with high probability

Compressed sensing: we want to create projection matrix Π such that for *any* x we can create from Πx a good k-sparse approximation to x

Doable! With Π that has $O(k \log(n/k))$ rows. Creating Π requires randomization but once found it can be used. Called RIP matrices. First due to Candes, Romberg, Tao and Donoho. Lot of work in signal processing and algorithms.