Frequency moments and Counting Distinct Elements

Lecture 05

September 8, 2020

Part I

Frequency Moments

Streaming model

- The input consists of m objects/items/tokens $e_1, e_2, \ldots, e_m$ that are seen one by one by the algorithm.
- The algorithm has "limited" memory say for B tokens where $B < m$ (often $B \ll m$) and hence cannot store all the input
- Want to compute interesting functions over input

Examples:

- Each token in a number from $[n]$
- High-speed network switch: tokens are packets with source, destination IP addresses and message contents.
- Each token is an edge in graph (graph streams)
- Each token in a point in some feature space
- Each token is a row/column of a matrix

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- A fundamental class of problems
- Formally introduced in the seminal paper of Alon, Matias, Szegedy titled "The Space Complexity of Approximating the Frequency Moments" in 1999.

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Example: $n = 5$ and stream is $4, 2, 4, 1, 1, 1, 4, 5$

Frequency Moments

- Stream consists of $e_1, e_2, \ldots, e_m$ where each e_i is an integer in $[n]$. We know n in advance (or an upper bound)
- Given a stream let f_i denote the frequency of i or number of times i is seen in the stream
- Consider vector $\mathbf{f} = (f_1, f_2, \ldots, f_n)$
- For $k \geq 0$ the k'th frequency moment $\boxed{F_k = \sum_i f_i^k}$. We can also consider the ℓ_k norm of $\mathbf{f}$ which is $\underline{(F_k)}^{1/k}$.

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$m = 8$ $\quad f_1 = 3 \quad f_2 = 1 \quad f_3 = 0 \quad f_4 = 3 \quad f_5 = 1$

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- $2 < k < \infty$

Frequency Moments: Questions

Estimation

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Questions easy if we have memory $\Omega(n)$: store $\mathbf{f}$ explicitly.
Interesting when memory is $\ll n$. Ideally want to do it with $\log^c n$ memory for some fixed $c \geq 1$ (polylog(n)). Note that $\log n$ is roughly the memory required to store one token/number.

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Relative approximation

Let $g(\sigma)$ be a real-valued *non-negative* function over streams σ.

Definition

Let $\mathcal{A}(\sigma)$ be the real-valued output of a randomized streaming algorithm on stream σ. We say that $\mathcal{A}$ provides an (α, β) relative approximation for a real-valued function g if for all σ:

$$\Pr\left[\left|\frac{\mathcal{A}(\sigma)}{g(\sigma)} - 1\right| > \alpha\right] \leq \beta.$$

Our ideal goal is to obtain a (ϵ, δ)-approximation for any given $\epsilon, \delta \in (0, 1)$.

Additive approximation

Let $g(\sigma)$ be a real-valued function over streams σ. If $g(\sigma)$ can be negative, focus on additive approximation.

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Let $\mathcal{A}(\sigma)$ be the real-valued output of a randomized streaming algorithm on stream σ. We say that $\mathcal{A}$ provides an (α, β) additive approximation for a real-valued function g if for all σ:

$$\Pr\left[|\mathcal{A}(\sigma) - g(\sigma)| > \alpha\right] \leq \beta.$$

When working with additive approximations some normalization/scaling is typically necessary. Our ideal goal is to obtain a (ϵ, δ)-approximation for any given $\epsilon, \delta \in (0, 1)$.

Part II

Estimating Distinct Elements

Distinct Elements

Given a stream σ how many distinct elements did we see?

Example: in a network switch, during some time window how many distinct destination (or source) IP addresses were seen in the packets?

1, 1, 1, 1, , -- 1

1, 10, 1, 1, 1, 1, 5, 5, 1, 1, 1, .., 1

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Offline solution?

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Offline solution? via Dictionary data structure

Offline Solution

DistinctElements

```
    Initialize dictionary D to be empty
    k ← 0
    While (stream is not empty) do
        Let e be next item in stream
        If (e ∉ D) then
            Insert e into D
            k ← k + 1
    EndWhile
    Output k
```

Offline Solution

DistinctElements

Initialize dictionary $\mathcal{D}$ to be empty
$k \leftarrow 0$
While (stream is not empty) do
    Let e be next item in stream
    If ($e \notin \mathcal{D}$) then
        Insert e into $\mathcal{D}$
        $k \leftarrow k + 1$
EndWhile
Output k

Which dictionary data structure?

Offline Solution

```
DistinctElements
    Initialize dictionary 𝒟 to be empty
    k ← 0
    While (stream is not empty) do
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        If (e ∉ 𝒟) then
            Insert e into 𝒟
            k ← k + 1
    EndWhile
    Output k
```

Which dictionary data structure?

- Binary search trees: space $O(k)$ and total time $O(m \log k)$
- Hashing: space $O(k)$ and expected time $O(m)$.

Hashing based idea

- Use hash function $h : [n] \to [N]$ for some N polynomial in n.
- Store only the minimum hash value seen. That is $\min_{e_i} h(e_i)$. Need only $O(\log n)$ bits since numbers are in range $[N]$.

$\log N \simeq (\log n)$

Hashing based idea

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Question: why is this good?

- Assume *idealized* hash function: $h : [n] \rightarrow [0, 1]$ that is fully random over the real interval

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- Suppose there are k distinct elements in the stream

$\underline{1}, \underline{1}, 100, 5, \underline{1}, 5, \underline{2}, 100, 5.$

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Question: why is this good?

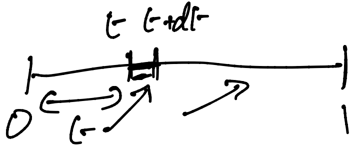
- Assume *idealized* hash function: $h : [n] \to [0, 1]$ that is fully random over the real interval
- Suppose there are k distinct elements in the stream
- What is the expected value of the minimum of hash values?

Analyzing idealized hash function

Lemma

Suppose $X_1, X_2, \ldots, X_k$ *are random variables that are independent and uniformaly distributed in* $[0,1]$ *and let* $Y = \min_i X_i$. *Then* $\mathbf{E}[Y] = \frac{1}{(k+1)}$.

$$\Pr\left[Y \in [t, t+dt] \right] = \binom{k}{1} \cdot (1-t)^{(k-1)} \, dt$$



$$E[Y] = \int_0^1 t \cdot \binom{k}{1} (1-t)^{k-1} \, dt = k \int_0^1 y^{k-1}(1-y) \, dy$$

$(1-y)$ $\quad$ y

$$= \frac{1}{k+1}.$$

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DistinctElements

```
Assume ideal hash function h : [n] → [0, 1]
y ← 1
While (stream is not empty) do
    Let e be next item in stream
    y ← min(y, h(e))
EndWhile
Output 1/y − 1
```

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Suppose $X_1, X_2, \ldots, X_k$ *are random variables that are independent and uniformaly distributed in* $[0,1]$ *and let* $Y = \min_i X_i$. *Then* $\mathbf{E}\left[Y^2\right] = \frac{2}{(k+1)(k+2)}$ *and* $Var(Y) = \frac{k}{(k+1)^2(k+2)} \leq \frac{1}{(k+1)^2}$.

$$E[Y^2] = \int_0^1 t^2 \cdot \underbrace{\binom{k}{1}(1-t)^{k-1}}\,dt$$

Analyzing idealized hash function

Apply standard methodology to go from exact statistical estimator to good bounds:

- average h parallel and independent estimates to reduce variance
- apply Chebyshev to show that the average estimator is a $(1+\epsilon)$-approximation with constant probability
- use preceding and median trick with $O(\log 1/\delta)$ parallel copies to obtain a $(1+\epsilon)$-approximation with probability $(1-\delta)$

Averaging and reducing variance

1. Run basic estimator independently and in parallel h times to obtain $X_1, X_2, \ldots, X_h$
2. Let $Z = \frac{1}{h} X_i$
3. Output $\frac{1}{Z} - 1$

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Claim: $\mathbf{E}[Z] = \frac{1}{(k+1)}$ and $Var(Z) \le \frac{1}{h}\frac{1}{(k+1)^2}$.

Choosing $h = 1/(\eta\epsilon^2)$ and using Chebyshev:
$\Pr\left[|Z - \frac{1}{k+1}| \ge \frac{\epsilon}{k+1}\right] \le \eta$.

$$\frac{\frac{1}{h}\cdot\frac{1}{\cancel{(k+1)^2}}}{\frac{\epsilon^2}{\cancel{(k+1)^2}}} \le \frac{\eta\cancel{\epsilon^2}}{\cancel{\epsilon^2}} \le \eta.$$

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Hence $\mathbf{Pr}\left[|(\frac{1}{Z} - 1) - k|\right] \geq O(\epsilon)k \leq \eta$.

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1 estimator
1 #

$\left(\frac{c}{\epsilon^2} \log \frac{1}{\delta}\right) \cdot \#s$

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Hence $\mathbf{Pr}\left[|(\frac{1}{Z} - 1) - k|\right] \geq O(\epsilon)k \leq \eta$.

Repeat $O(\log 1/\delta)$ times and output median. Error probability $< \delta$.

Algorithm via regular hashing

Do not have idealized hash function.

- Use $h : [n] \to [N]$ for appropriate choice of N
- Use pairwise independent hash family $\mathcal{H}$ so that random $h \in \mathcal{H}$ can be stored in small space and computation can be done in small memory and fast

Several variants of idea with different trade offs between

- memory
- time to process each new element of the stream
- approximation quality and probability of success