Limited independence and Hashing

Lecture 05/06

September 8 and 10, 2020

Pseudorandomness

Randomized algorithms rely on independent random bits

Psuedorandomness: when can we *avoid* or *limit* number of random bits?

- Motivated by fundamental theoretical questions and applications
- Applications: hashing, cryptography, streaming, simulations, derandomization, ...
- A large topic in TCS with many connections to mathematics.

This course: need t-wise independent variables and hashing

Part I

Pairwise and t-wise independent random variables

Pairwise independent random variables

Definition

Discrete random variables $X_1, X_2, \ldots, X_n$ from a range B are independent if for all $b_1, b_2, \ldots, b_n \in B$

$$\Pr[X_1 = b_1, X_2 = b_2, \ldots, X_n = b_n] = \prod_{i=1}^{n} \Pr[X_i = b_i].$$

Uniformly distributed if $\Pr[X_i = b] = 1/|B|$ for all $i, b \in B$.

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Definition

Random variables $X_1, X_2, \ldots, X_n$ from a range B are **pairwise** independent if for all $1 \leq i < j \leq n$ and for all $b, b' \in B$,

$$\Pr[X_i = b, X_j = b'] = \Pr[X_i = b] \cdot \Pr[X_j = b'] \,.$$

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Example: X_1, X_2 are independent bits (variables from $\{0, 1\}$) and $X_3 = X_1 \oplus X_2$. X_1, X_2, X_3 are pairwise independent but not independent.

t-wise independence

Generalizing pairwise independence:

Definition

Random variables $X_1, X_2, \ldots, X_n$ from a range B are t-wise independent for integer $t > 1$ $X_{i_1}, X_{i_2}, \ldots, X_{i_t}$ are independent for any $i_1 \neq i_2 \neq \ldots \neq i_t \in \{1, 2, \ldots, n\}$.

As t increases the variables become more and more independent. If $t = n$ the variables are independent.

Motivation for pairwise/t-wise independence from streaming

Want n uniformly distr random variables $X_1, X_2, \ldots, X_n$, say bits
But cannot store n bits because n is too large.

Achievable:

- storage of $O(\log n)$ random bits
- given i where $1 \leq i \leq n$ can generate X_i in $O(\log n)$ time
- $X_1, X_2, \ldots, X_n$ are pairwise independent and uniform
- Hence, with small storage, can generate n random variables "on the fly". In several applications, pairwise independence (or generalizations) suffice

Generating pairwise independent bits

Assume for simplicity $n = 2^k - 1$ (otherwise consider nearest power of 2). Hence $k = O(\log n)$

- Let $Y_1, Y_2, \ldots, Y_k$ be independent bits
- For any $S \subset \{1, 2, \ldots, k\}$, $S \neq \emptyset$, define $X_S = \oplus_{i \in S} Y_i$
- $2^k - 1$ random variables X_S

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Proof.

X_S and X_T are both uniformaly distributed over $\{0, 1\}$. Suppose $S - T \neq \emptyset$. Even knowing all outcomes of variables in T the variables in $S - T$ are independent and hence $\Pr[X_S = 0 \mid T] = 1/2$ and hence X_S is independent of X_T. If $S \subset T$ then apply same argument to $T - S$. □

Pairwise independent variables with larger range

Suppose we want n pairwise independent random variables in range $\{0, 1, 2, \ldots, m-1\}$ where $m = 2^k - 1$ for some k

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Suppose we want n pairwise independent random variables in range $\{0, 1, 2, \ldots, m-1\}$ where $m = 2^k - 1$ for some k

- Now each X_i needs to be a $\log m$ bit string
- Use preceding construction for each bit independently
- Requires $O(\log m \log n)$ bits total
- Can in fact do $O(\log n + \log m)$ bits

Using prime numbers and fields

Assume $n = m = p$ where p is a prime number

Want p pairwise random variables distributed uniformly in $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$

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- Choose $a, b \in \{0, 1, 2, \ldots, p-1\}$ uniformly and independently at random. Requires $2\lceil \log p \rceil$ random bits
- For $0 \leq i \leq p-1$ set $X_i = ai + b \mod p$
- Note that one needs to store only a, b, p and can generate X_i efficiently on the fly from i

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- Note that one needs to store only a, b, p and can generate X_i efficiently on the fly from i

Exercise: Prove that each X_i is uniformly distributed in $\mathbb{Z}_p$.
Claim: For $i \neq j$, X_i and X_j are independent.

Using prime numbers and fields

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Some math required:

- $\mathbb{Z}_p$ is a field for any prime p. That is $\{0, 1, 2, \ldots, p-1\}$ forms a commutative group under addition mod p (easy). And more importantly $\{1, 2, \ldots, p-1\}$ forms a commutative group under multiplication.

Some math required...

Lemma (LemmaUnique)

Let p be a prime number,
x: an integer number in $\{1, \ldots, p-1\}$.
$\implies$ *There exists a unique y s.t. $xy = 1 \mod p$.*

In other words: For every element there is a unique inverse.
$\implies \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ when working modulo p is a *field*.

Proof of LemmaUnique

Claim

Let p be a prime number. For any $x, y, z \in \{1, \ldots, p-1\}$ s.t. $y \neq z$, we have that $xy \mod p \neq xz \mod p$.

Proof.

Assume for the sake of contradiction $xy \mod p = xz \mod p$.

$$\begin{aligned} x(y-z) &= 0 \mod p \\ \implies & \quad p \text{ divides } \; x(y-z) \\ \implies & \quad p \text{ divides } \; y-z \\ \implies & \quad y-z=0 \\ \implies & \quad y=z. \end{aligned}$$

And that is a contradiction. $\square$

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By the above claim if $xy = 1 \mod p$ and $xz = 1 \mod p$ then $y = z$. Hence uniqueness follows.

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Existence. For any $x \in \{1, \ldots, p-1\}$ we have that $\{x * 1 \mod p, x * 2 \mod p, \ldots, x * (p-1) \mod p\} = \{1, 2, \ldots, p-1\}$.
$\implies$ There exists a number $y \in \{1, \ldots, p-1\}$ such that $xy = 1 \mod p$.

$\square$

Proof of pairwise independence

Lemma

If $i \neq j$ *then for each* $(r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p$ *there is exactly* **one** *pair* $(a, b) \in \mathbb{Z}_p \times \mathbb{Z}_p$ *such that*

$$ai + b \mod p = r \text{ and } aj + b \mod p = s.$$

Proof.

Solve the two equations:

$$ai + b = r \mod p \qquad \text{and} \qquad aj + b = s \mod p$$

We get $a = \frac{r-s}{i-j} \mod p$ and $b = r - ax \mod p$. □

One-to-one correspondence between (a, b) and (r, s)

Proof of pairwise independence

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One-to-one correspondence between (a, b) and (r, s)
$\Rightarrow$ if (a, b) is uniformly at random from $\mathbb{Z}_p \times \mathbb{Z}_p$ then (r, s) is uniformly at random from $\mathbb{Z}_p \times \mathbb{Z}_p$. X_i, X_j independent.

Pairwise independence for n, m powers of 2

We saw how to create n pairwise independent random variables when $n = m = p$ where p is a prime number. We want n, m arbitrary. Easy to assume n is power of 2 (discard the unnecessary rvs) but harder if m is not power of 2. Here we only consider powers of 2.

$n > m$ is the more difficult case and also relevant.

The following is a fundamental theorem on finite fields.

Theorem

Every finite field $\mathbb{F}$ has order p^k for some prime p and some integer $k \geq 1$. For every prime p and integer $k \geq 1$ there is a finite field $\mathbb{F}$ of order p^k and is unique up to isomorphism.

We will assume n and m are powers of 2. From above can assume we have a field $\mathbb{F}$ of size $n = 2^k$.

Pairwise independence for n, m powers of 2

We have a field $\mathbb{F}$ of size $n = 2^k$.

Generate n pairwise independent random variables from $[n]$ to $[n]$ by picking random $a, b \in \mathbb{F}$ and setting $X_i = ai + b$ (operations in $\mathbb{F}$). From previous proof (we only used that $\mathbb{Z}_p$ is a field) X_i are pairwise independent.

Now $X_i \in [n]$. Truncate X_i to $[m]$ by dropping the most significant $\log n - \log m$ bits. Resulting variables are still pairwise independent (both n, m being powers of 2 useful here).

Need to only store a, b, n and can generate $X_i = ai + b$. Skipping details on computational aspects of $\mathbb{F}$ which are closely tied to the proof of the theorem on fields.

t-wise independence

Generalizing pairwise independence:

Definition

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As t increases the variables become more and more independent. If $t = n$ the variables are independent.

Fact: For any n, m one can create n random t-wise independent random variables from the range $[m]$ using $O(t(\log n + \log m))$ true random bits. Can store only bits and generate the variables on the fly in $O(t\text{polylog}(m + n))$ time.

t-wise independence

Construction using polynomials

- Let $\mathbb{F}$ be a field
- Pick t random (with replacement) numbers from $\mathbb{F}$: $a_0, a_1, \ldots, a_{t-1}$
- For each $i \in [|\mathbb{F}|]$ set $X_i = a_0 + a_1 i + a_2 i^2 + \ldots + a_{t-1} i^{t-1}$

Pairwise Independence and Chebyshev's Inequality

Chebyshev's Inequality

For $a \geq 0$, $\Pr[|X - \mathbf{E}[X]| \geq a] \leq \frac{Var(X)}{a^2}$ equivalently for any $t > 0$, $\Pr[|X - \mathbf{E}[X]| \geq t\sigma_X] \leq \frac{1}{t^2}$ where $\sigma_X = \sqrt{Var(X)}$ is the standard deviation of X.

Suppose $X = X_1 + X_2 + \ldots + X_n$.
If $X_1, X_2, \ldots, X_n$ are independent then $Var(X) = \sum_i Var(X_i)$.
Recall application to random walk on line

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Lemma

Suppose $X = \sum_i X_i$ and $X_1, X_2, \ldots, X_n$ are pairwise independent, then $Var(X) = \sum_i Var(X_i)$.

Part II

Hashing

Balls and Bins and Load Balancing

Suppose we want to distribute jobs to machines in a simple way to achieve load balancing.

Throwing each new job into a random machine is a simple, distributed, oblivious strategy with many benefits

Balls and bins is simple mathematical model to analyze the core principles

Balls and Bins $\rightarrow$ Hashing

Hashing:

- Want a "function" $h : \mathcal{U} \rightarrow B$.
- Want h to behave like a "random function". That is for any distinct $x_1, x_2, \ldots, x_n \in \mathcal{U}$ we have $h(x_1), h(x_2), \ldots, h(x_n)$ to be uniformly distributed over B and independent.
- But want h to be efficiently computable and stored in small memory

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Many applications: hash tables as dictionary data structure, cryptography/security, pseudorandomness, $\ldots$

Dictionary Data Structure

1. $\mathcal{U}$: universe of keys : numbers, strings, images, etc.
2. Data structure to store a subset $S \subseteq \mathcal{U}$
3. **Operations:**
   1. **Search/look up**: given $x \in \mathcal{U}$ is $x \in S$?
   2. **Insert**: given $x \notin S$ add x to S.
   3. **Delete**: given $x \in S$ delete x from S
4. **Static** structure: S given in advance or changes very infrequently, main operations are lookups.
5. **Dynamic** structure: S changes rapidly so inserts and deletes as important as lookups.

Dictionary Data Structure

- Standard dictionary data structures such binary search trees rely on universe $\mathcal{U}$ being a total order and hence can be compared
- Comparison based data structures take $\Theta(\log n)$ comparisons when storing n items from $\mathcal{U}$ and typically require pointer based data structure
- All objects represented in computers are essentially strings so technically one can use a comparison based data structure always
- Disadvantages of comparison based data structures:
  - Comparisons are expensive for many objects
  - Dynamic memory allocation and pointers
- Hashing based dictionaries:
  - $O(1)$ expected time operations
  - Depending on implementation, can avoid pointers

Hashing and Hash Tables

Hash Table data structure:

1. A (hash) table/array T of size m (the table **size**).
2. A hash function $h : \mathcal{U} \to \{0, \ldots, m-1\}$.
3. Item $x \in \mathcal{U}$ hashes to slot $h(x)$ in T.

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Ideal situation:

1. Each element $x \in S$ hashes to a distinct slot in T. Store x in slot $h(x)$
2. **Lookup**: Given $y \in \mathcal{U}$ check if $T[h(y)] = y$. $O(1)$ time!

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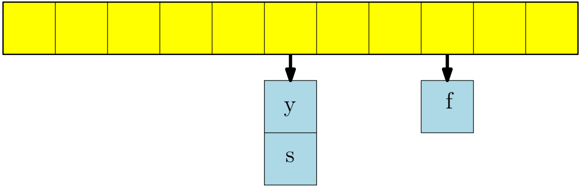
Collisions unavoidable if $|T| < |\mathcal{U}|$. Several techniques to handle them.

Handling Collisions: Chaining

Collision: $h(x) = h(y)$ for some $x \neq y$.

Chaining/Open hashing to handle collisions:

1. For each slot i store all items hashed to slot i in a linked list. $T[i]$ points to the linked list
2. **Lookup**: to find if $y \in \mathcal{U}$ is in T, check the linked list at $T[h(y)]$. Time proportion to size of linked list.



Chain length determines time for operations. Ideally want $O(1)$.

Hash Functions

Parameters: $N = |\mathcal{U}|$ (very large), $m = |T|$, $n = |S|$
Goal: $O(1)$-time lookup, insertion, deletion.

Single hash function

If $N \geq m^2$, then for any hash function $h : \mathcal{U} \rightarrow T$ there exists $i < m$ such that at least $N/m \geq m$ elements of $\mathcal{U}$ get hashed to slot i.

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Such a bad set may lead to $O(m)$ lookup time!

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Such a bad set may lead to $O(m)$ lookup time!

In practice:

- Dictionary applications: choose a simple hash function and hope that worst-case bad sets do not arise
- Crypto applications: create "hard" and "complex" function very carefully which makes finding collisions difficult

Hashing from a theoretical point of view

- Consider a family $\mathcal{H}$ of hash functions with *good properties* and choose h randomly from $\mathcal{H}$
- Guarantees: small # collisions in expectation for any given S.
- $\mathcal{H}$ should allow efficient sampling.
- Each $h \in \mathcal{H}$ should be efficient to evaluate and require small memory to store.

In other worse a hash function is a "pseudorandom" function

Strongly Universal Hashing

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1. **Uniform:** Consider any element $x \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then x should go into a random slot in T. In other words $\Pr[h(x) = i] = 1/m$ for every $0 \leq i < m$.

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2. **(2)-Strongly Universal**: Consider any two distinct elements $x, y \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then $h(x)$ and $h(y)$ should be independent random variables.

Strongly Universal Hashing

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2. **(2)-Strongly Universal**: Consider any two distinct elements $x, y \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then $h(x)$ and $h(y)$ should be independent random variables.

Note: Fix $x \in \mathcal{U}$. $h(x)$ is a *random variable* with range $\{0, 1, 2, \ldots, m-1\}$. Strong universal hash family implies that the variables $h(x), x \in S$ are uniform and pairwise independent random variables.

Universal Hashing

Question: What are good properties of $\mathcal{H}$ in distributing data?

- **(2)-Universal:** Consider any two distinct elements $x, y \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then the probability of a collision between x and y should be at most $1/m$. In other words $\Pr[h(x) = h(y)] \leq 1/m$.

Note: we do not insist on uniformity.

(Strongly) Universal Hashing

Definition

A family of hash functions $\mathcal{H}$ is (2-)**strongly universal** if for all distinct $x, y \in \mathcal{U}$, $h(x)$ and $h(y)$ are independent for h chosen uniformly at random from $\mathcal{H}$, and for all x, $h(x)$ is uniformly distributed.

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Generalizes to t-strongly universal and t-universal families. Need property for any tuple of t items.

Analyzing Universal Hashing

Question: Fixing set S, what is the *expected* time to look up $x \in S$ when h is picked uniformly at random from $\mathcal{H}$?

1. $\ell(x)$ **:** the size of the list at $T[h(x)]$. We want $\mathrm{E}[\ell(x)]$
2. For $y \in S$ let $D_y = 1$ if $h(y) = h(x)$, else 0.
   $\ell(x) = \sum_{y \in S} D_y$

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$$
\begin{aligned}
\mathbf{E}[\ell(x)] &= \sum_{y \in S} \mathrm{E}[D_y] = \sum_{y \in S} Pr[h(x) = h(y)] \\
&\leq 1 + \sum_{y \in S, y \neq x} \frac{1}{m} \qquad (\mathcal{H} \text{ is a universal hash family}) \\
&\leq 1 + (|S| - 1)/m \leq 2 \quad \text{if } |S| \leq m
\end{aligned}
$$

Analyzing Universal Hashing

Question: What is the *expected* time to look up x in T using h assuming chaining used to resolve collisions?

Answer: $O(n/m)$.

Analyzing Universal Hashing

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Answer: $O(n/m)$.

Comments:

1. $O(1)$ expected time also holds for insertion.
2. Analysis assumes static set S but holds as long as S is a set formed with at most $O(m)$ insertions and deletions.
3. **Worst-case**: look up time can be large! How large? In principle $\Omega(n)$ time but if $\mathcal{H}$ has good properties then $O(\sqrt{n})$ or $O(\log n / \log\log n)$ with high probability.

Universal Hash Family

Universal: $\mathcal{H}$ such that $\Pr[h(x) = h(y)] = 1/m$.

All functions

$\mathcal{H}$: Set of all possible functions $h : \mathcal{U} \to \{0, \ldots, m-1\}$.

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We need *compactly representable* universal family.

Compact Stongly Universal Hash Family

Similar to construction of N pairwise independent random variables with range $[m]$.

The function is given by the algorithm to construct X_i given i.

Can do with $O(\log N)$ bits of storage since $N \geq m$ in hashing application.

A Compact Universal Hash Family

Parameters: $N = |\mathcal{U}|$, $m = |T|$, $n = |S|$. Assumption $m \leq N$.

1. Choose a **prime** number $p \geq N$. $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ is a field.
2. For $a, b \in \mathbb{Z}_p$, $a \neq 0$, define the hash function $h_{a,b}$ as $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.
3. Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0\}$. Note that $|\mathcal{H}| = p(p-1)$.

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$\mathcal{H}$ is a universal hash family.

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Theorem

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Comments:

1. Hash family is of small size, easy to sample from.
2. Easy to store a hash function (a, b have to be stored) and evaluate it.

A Compact Universal Hash Family

- $g(x) = ax + b$ is uniformly distributed in $\{0, 1, \ldots, p-1\}$ but $h(x)$ is not uniformly distributed unless $m = p$.
- $\Pr[h(x) = i] \leq 2/m$ for any i.

Bloom Filters

Hashing:

1. To insert x in dictionary store x in table in location $h(x)$
2. To lookup y in dictionary check contents of location $h(y)$

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Bloom Filter: tradeoff space for false positives

1. Storing items in dictionary expensive in terms of memory, especially if items are unwieldy objects such a long strings, images, etc with *non-uniform* sizes.
2. To insert x in dictionary set *bit* to $\mathbf{1}$ in location $h(x)$ (initially all bits are set to $\mathbf{0}$)
3. To lookup y if bit in location $h(y)$ is $\mathbf{1}$ say yes, else no.

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Bloom Filter: tradeoff space for false positives

1. To insert x in dictionary set *bit* to $\mathbf{1}$ in location $h(x)$ (initially all bits are set to $\mathbf{0}$)
2. To lookup y if bit in location $h(y)$ is $\mathbf{1}$ say yes, else no
3. No false negatives but false positives possible due to collisions

Reducing false positives:

1. Pick k hash functions $h_1, h_2, \ldots, h_k$ *independently*
2. To insert x, for each i, set bit in location $h_i(x)$ in table i to $\mathbf{1}$
3. To lookup y compute $h_i(y)$ for $1 \leq i \leq k$ and say yes only if each bit in the corresponding location is $\mathbf{1}$, otherwise say no. If probability of false positive for one hash function is $\alpha < 1$ then with k independent hash function it is α^k.

Take away points

1. Hashing is a powerful and important technique for dictionaries. Many practical applications.
2. Randomization fundamental to understanding hashing.
3. Good and efficient hashing possible in theory and practice with proper definitions (universal, perfect, etc).
4. Related ideas of creating a compact fingerprint/sketch for objects is very powerful in theory and practice.

Practical Issues

Hashing used typically for integers, vectors, strings etc.

- Universal hashing is defined for integers. To implement for other objects need to map objects in some fashion to integers (via representation)
- Practical methods for various important cases such as vectors, strings are studied extensively. See http://en.wikipedia.org/wiki/Universal_hashing for some pointers.
- Details on Cuckoo hashing and its advantage over chaining http://en.wikipedia.org/wiki/Cuckoo_hashing.
- Recent important paper bridging theory and practice of hashing. "The power of simple tabulation hashing" by Mikkel Thorup and Mihai Patrascu, 2011. See http://en.wikipedia.org/wiki/Tabulation_hashing