NP-Completeness of 3-Color and SAT

Lecture 24

April 23, 2015

Recap

NP: languages that have non-deterministic polynomial time algorithms

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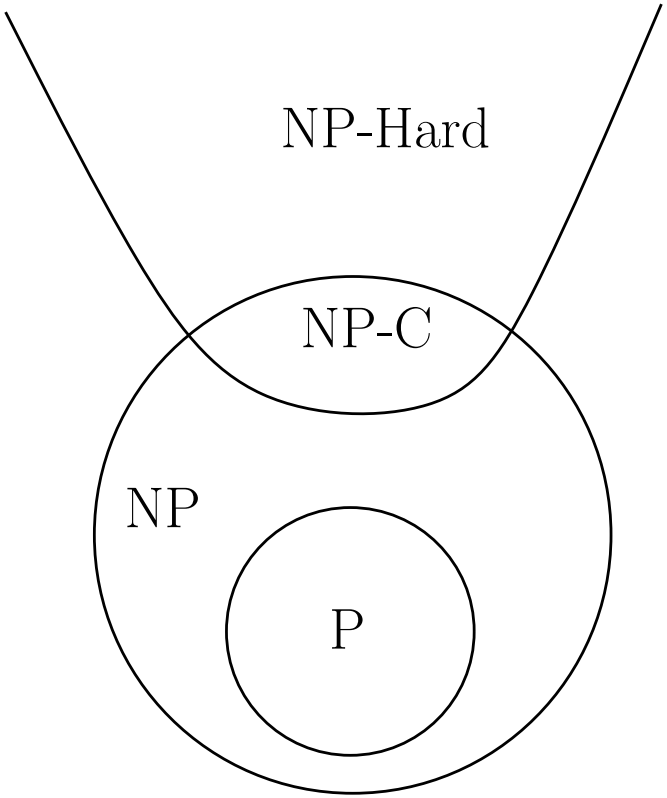
Theorem (Cook-Levin)

SAT *is* **NP-Complete**.

Pictorial View



P and NP

Possible scenarios:

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2. **P ≠ NP**

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Question: Suppose **P ≠ NP**. Is every problem in **NP** \ **P** also **NP-Complete**?

P and NP

Possible scenarios:

1. $\mathbf{P} = \mathbf{NP}$.
2. $\mathbf{P} \neq \mathbf{NP}$

Question: Suppose $\mathbf{P} \neq \mathbf{NP}$. Is every problem in $\mathbf{NP} \setminus \mathbf{P}$ also **NP-Complete**?

Theorem (Ladner)

If $\mathbf{P} \neq \mathbf{NP}$ then there is a problem/language $\mathbf{X} \in \mathbf{NP} \setminus \mathbf{P}$ such that $\mathbf{X}$ is not **NP-Complete**.

Part I

NP-Completeness of Graph Coloring

Graph Coloring

Problem: Graph Coloring

Instance: $G = (V, E)$: Undirected graph, integer **k**.
Question: Can the vertices of the graph be colored using **k** colors so that vertices connected by an edge do not get the same color?

Graph 3-Coloring

Problem: 3 Coloring

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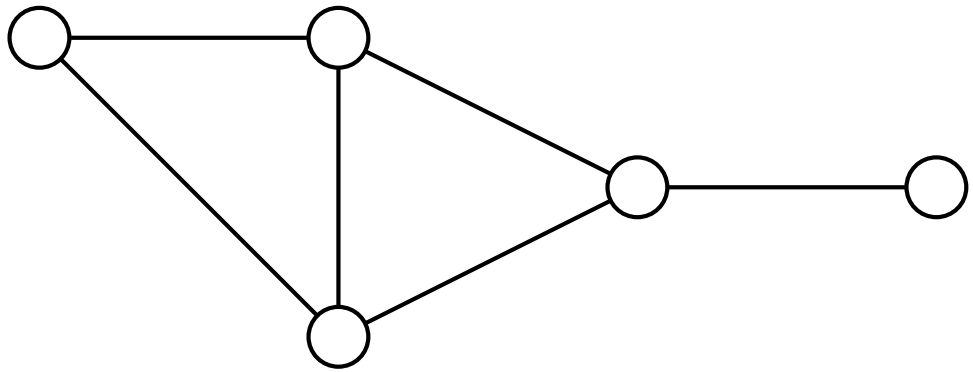

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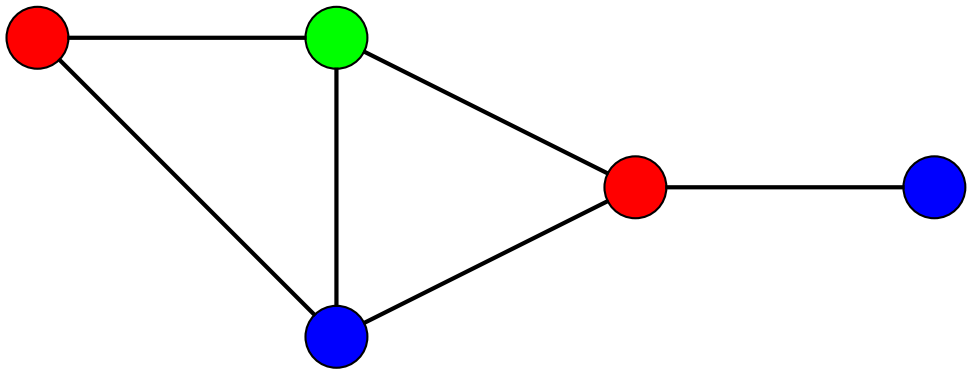

Graph Coloring

Observation: If **G** is colored with **k** colors then each color class (nodes of same color) form an independent set in **G**. Thus, **G** can be partitioned into **k** independent sets iff **G** is **k**-colorable.

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Graph **2**-Coloring can be decided in polynomial time.

G is **2**-colorable iff **G** is bipartite! There is a linear time algorithm to check if **G** is bipartite using **BFS**

Graph Coloring and Register Allocation

Register Allocation

Assign variables to (at most) **k** registers such that variables needed at the same time are not assigned to the same register

Interference Graph

Vertices are variables, and there is an edge between two vertices, if the two variables are "live" at the same time.

Observations

- [Chaitin] Register allocation problem is equivalent to coloring the interference graph with **k** colors
- Moreover, **3-COLOR $\leq_P$ k-Register Allocation**, for any **$k \geq 3$**

Class Room Scheduling

Given **n** classes and their meeting times, are **k** rooms sufficient?

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Reduce to Graph **k**-Coloring problem

Create graph **G**

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Exercise: **G** is **k**-colorable iff **k** rooms are sufficient

Frequency Assignments in Cellular Networks

Cellular telephone systems that use Frequency Division Multiple Access (FDMA) (example: GSM in Europe and Asia and AT&T in USA)

- Breakup a frequency range $[a, b]$ into disjoint *bands* of frequencies $[a_0, b_0], [a_1, b_1], \ldots, [a_k, b_k]$
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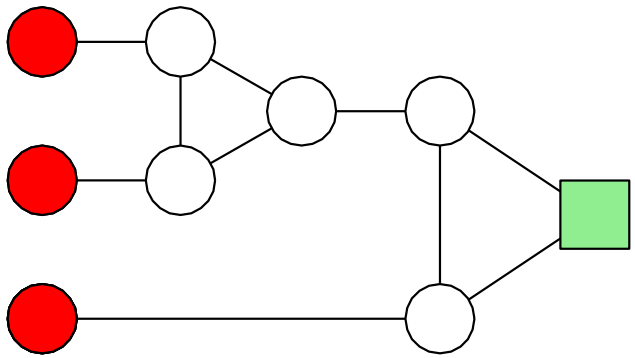
Problem: given k bands and some region with n towers, is there a way to assign the bands to avoid interference?

Can reduce to k-coloring by creating intereference/conflict graph on towers.

3 color this gadget.

You are given three colors: red, green and blue. Can the following graph be three colored in a valid way (assuming that some of the nodes are already colored as indicated).



(A) Yes.

(B) No.

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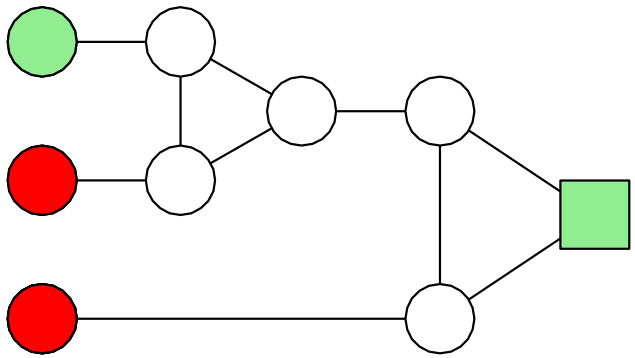

(A) Yes.

(B) No.

3-Coloring is NP-Complete

- **3-Coloring** is in **NP**.
  - Non-deterministically guess a 3-coloring for each node
  - Check if for each edge **(u, v)**, the color of **u** is different from that of **v**.
- Hardness: We will show **3-SAT** $\leq_P$ **3-Coloring**.

Reduction Idea

Start with **3SAT** formula (i.e., **3CNF** formula) φ with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. Create graph G_φ such that G_φ is 3-colorable iff φ is satisfiable

- need to establish truth assignment for $x_1, \ldots, x_n$ via colors for some nodes in G_φ.

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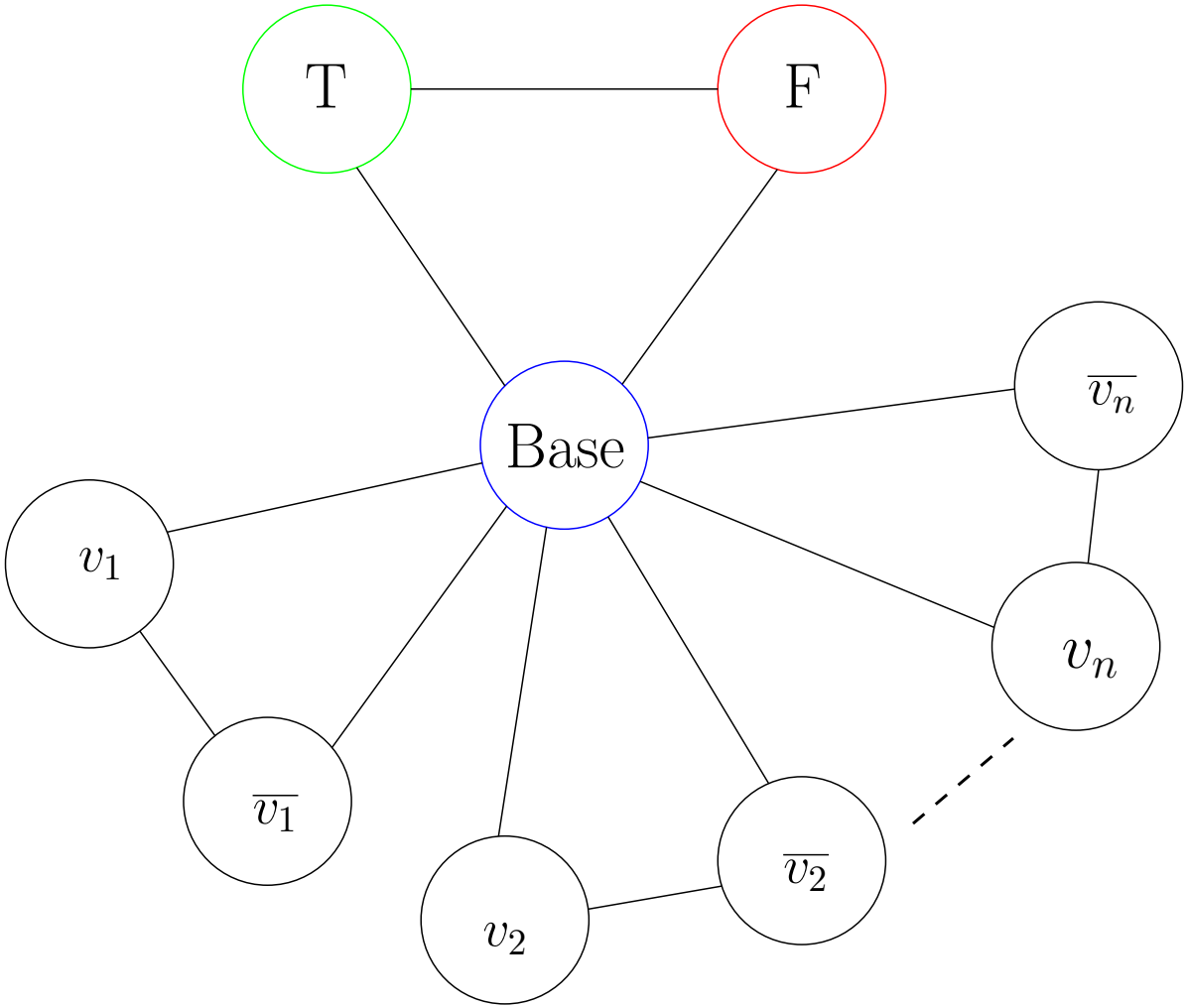
- Need to add constraints to ensure clauses are satisfied (next phase)

Figure

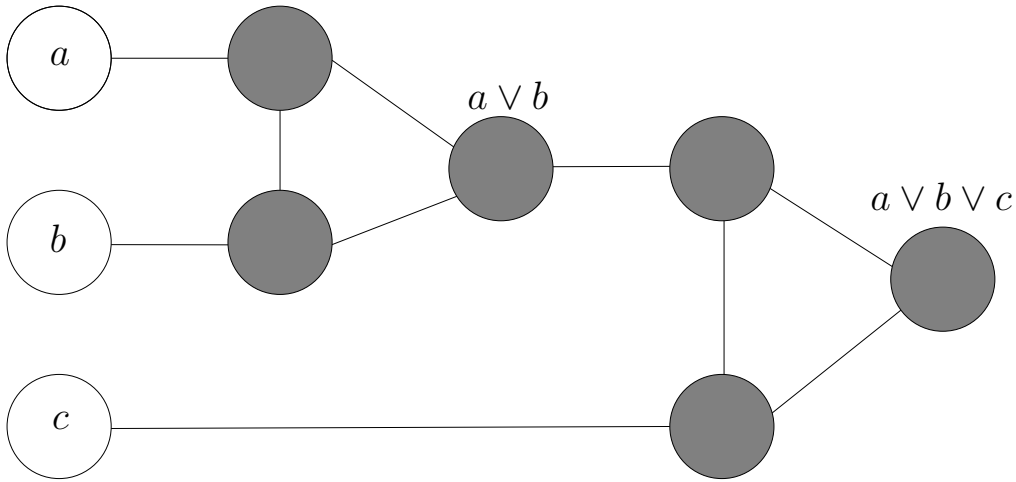


Clause Satisfiability Gadget

For each clause $\mathbf{C_j = (a \vee b \vee c)}$, create a small gadget graph

- gadget graph connects to nodes corresponding to **a, b, c**
- needs to implement OR

OR-gadget-graph:



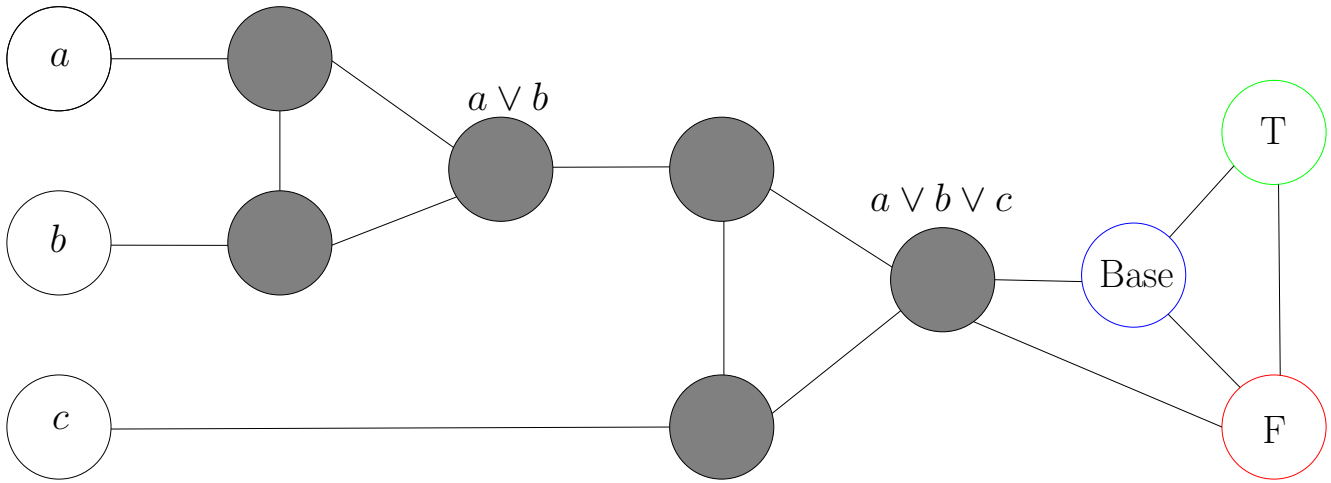
OR-Gadget Graph

Property: if **a, b, c** are colored False in a 3-coloring then output node of OR-gadget has to be colored False.

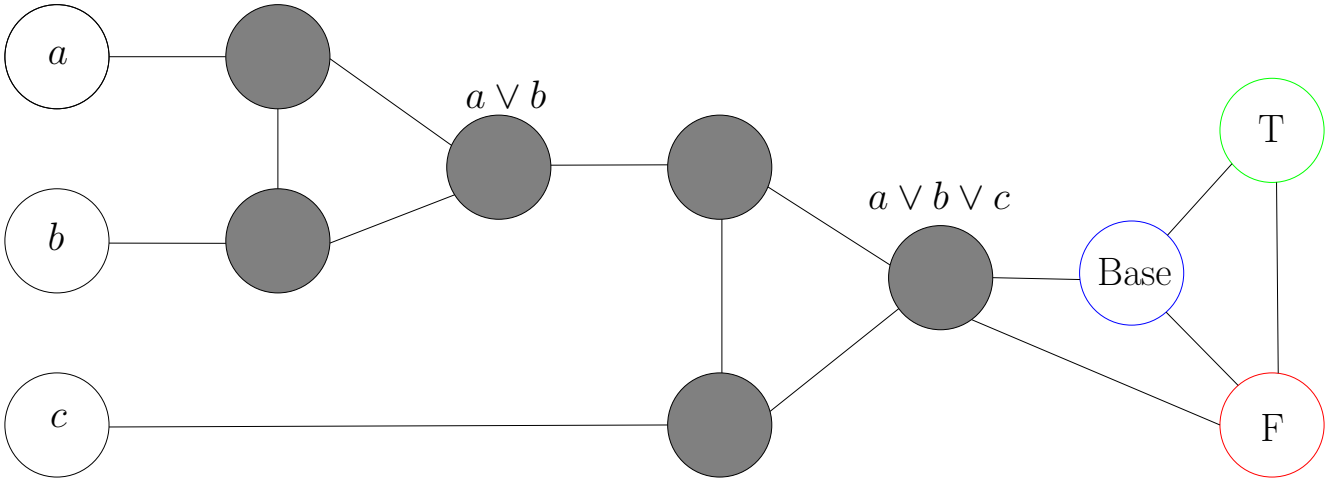
Property: if one of **a, b, c** is colored True then OR-gadget can be 3-colored such that output node of OR-gadget is colored True.

Reduction

- create triangle with nodes True, False, Base
- for each variable $\mathbf{x_i}$ two nodes $\mathbf{v_i}$ and $\mathbf{\bar{v}_i}$ connected in a triangle with common Base
- for each clause $\mathbf{C_j = (a \vee b \vee c)}$, add OR-gadget graph with input nodes $\mathbf{a, b, c}$ and connect output node of gadget to both False and Base



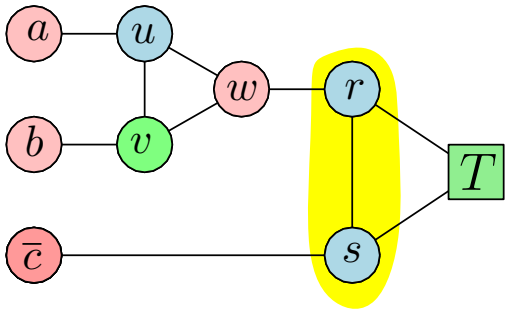
Reduction



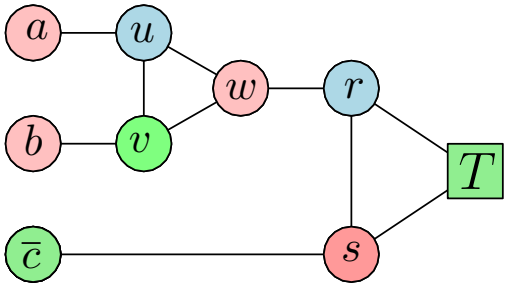
Claim

No legal **3***-coloring of above graph (with coloring of nodes* **T, F, B** *fixed) in which* **a, b, c** *are colored False. If any of* **a, b, c** *are colored True then there is a legal* **3***-coloring of above graph.*

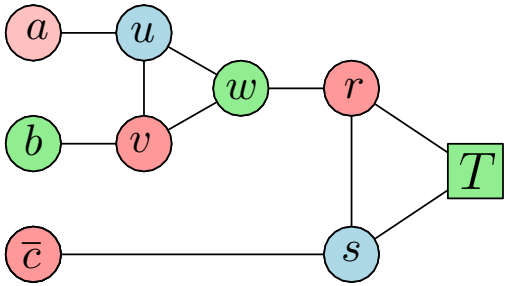
3 coloring of the clause gadget



FFF - **BAD**



FFT



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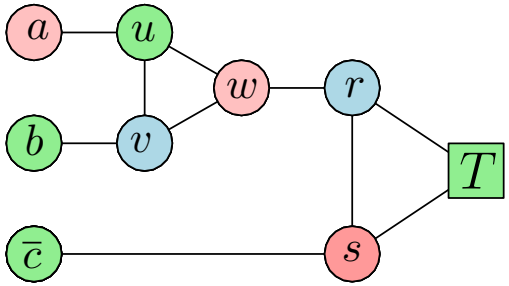
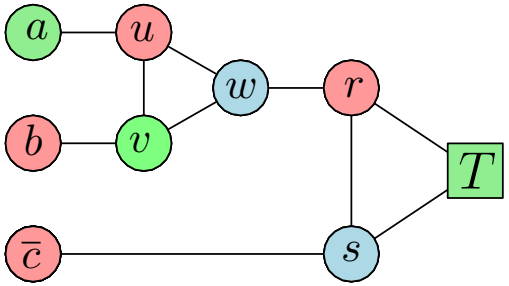
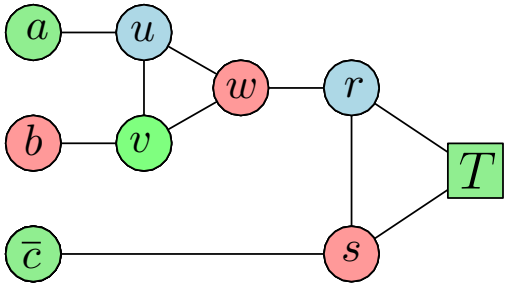


FTT



TFF



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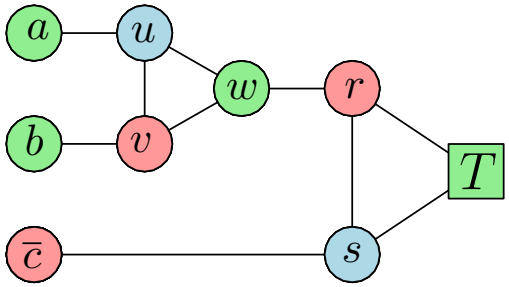

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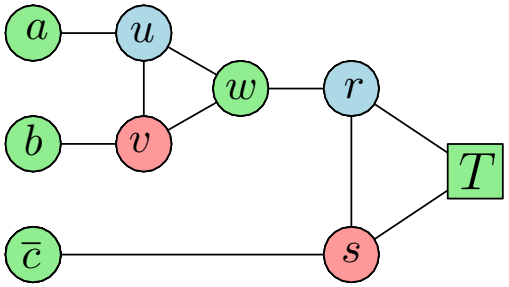
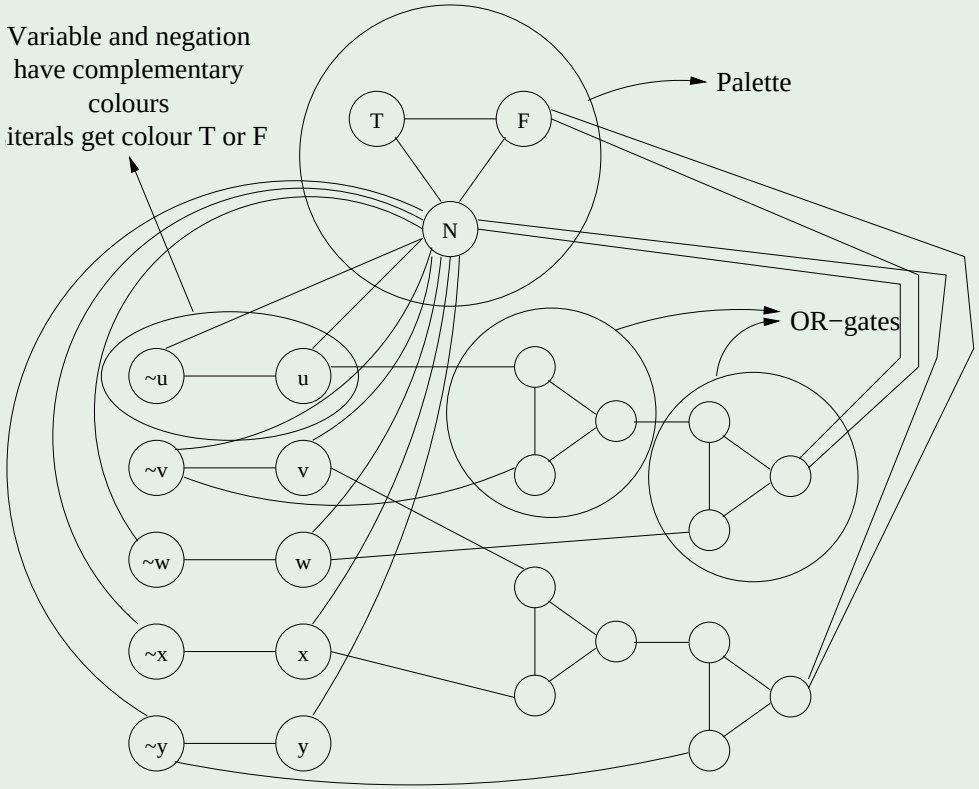


TTT

Reduction Outline

Example

$\varphi = (\mathbf{u} \vee \neg\mathbf{v} \vee \mathbf{w}) \wedge (\mathbf{v} \vee \mathbf{x} \vee \neg\mathbf{y})$



Correctness of Reduction

φ is satisfiable implies G_φ is 3-colorable

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- if v_i is colored True then set x_i to be True, this is a legal truth assignment

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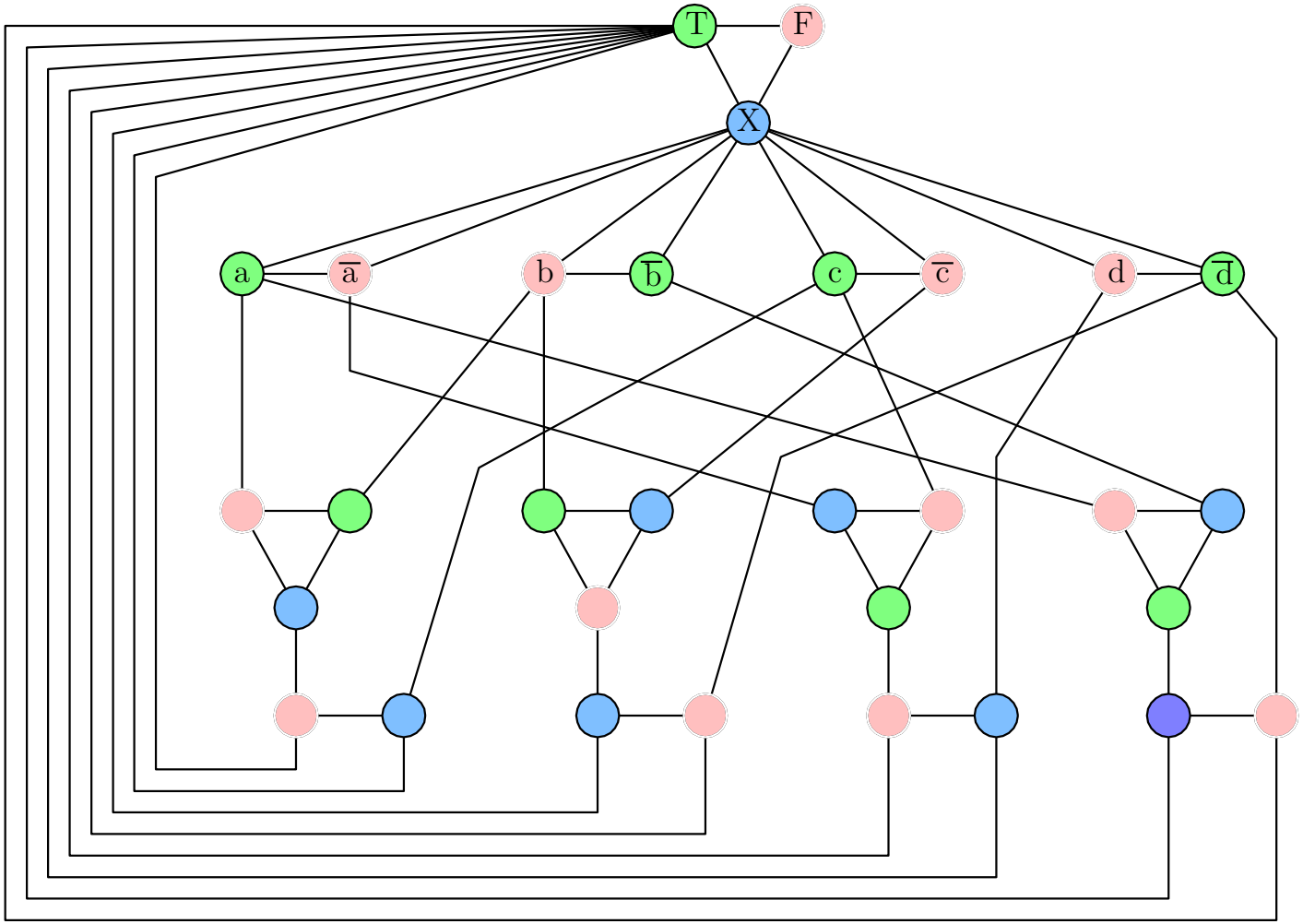
G_φ is 3-colorable implies φ is satisfiable

- if v_i is colored True then set x_i to be True, this is a legal truth assignment
- consider any clause $C_j = (a \vee b \vee c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!

Graph generated in reduction...

... from 3SAT to 3COLOR



Part II

Proof of Cook-Levin Theorem

Cook-Levin Theorem

Theorem (Cook-Levin)

SAT *is* **NP-Complete**.

We have already seen that **SAT** is in **NP**.

Need to prove that *every* language $\mathbf{L} \in \mathbf{NP}$, $\mathbf{L} \leq_{\mathbf{P}} \mathbf{SAT}$

Cook-Levin Theorem

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Need to prove that *every* language $\mathbf{L} \in \mathbf{NP}$, $\mathbf{L} \leq_{\mathbf{P}} \mathbf{SAT}$

Difficulty: Infinite number of languages in **NP**. Must *simultaneously* show a *generic* reduction strategy.

High-level Plan

What does it mean that $L \in NP$?
$L \in NP$ implies that there is a non-deterministic TM M and polynomial $p()$ such that

$$L = \{x \in \Sigma^* \mid M \text{ accepts } x \text{ in at most } p(|x|) \text{ steps}\}$$

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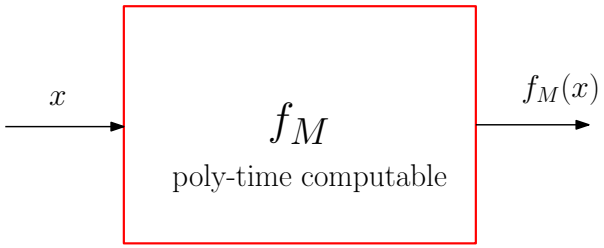
We will describe a reduction f_M that depends on M, p such that:

- f_M takes as input a string x and outputs a SAT formula $f_M(x)$
- f_M runs in time polynomial in $|x|$
- $x \in L$ if and only if $f_M(x)$ is satisfiable

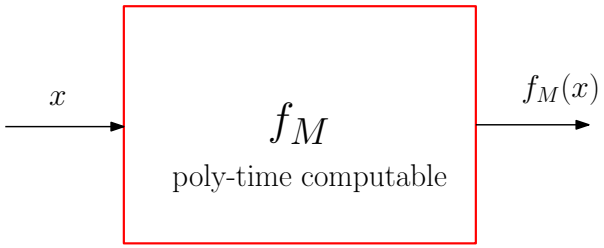
Plan continued



$f_M(x)$ is satisfiable if and only if **$x \in L$**

$f_M(x)$ is satisfiable if and only if non-det **M** accepts **x** in **$p(|x|)$** steps

Plan continued



$\mathbf{f_M(x)}$ is satisfiable if and only if $\mathbf{x \in L}$
$\mathbf{f_M(x)}$ is satisfiable if and only if non-det **M** accepts **x** in $\mathbf{p(|x|)}$ steps

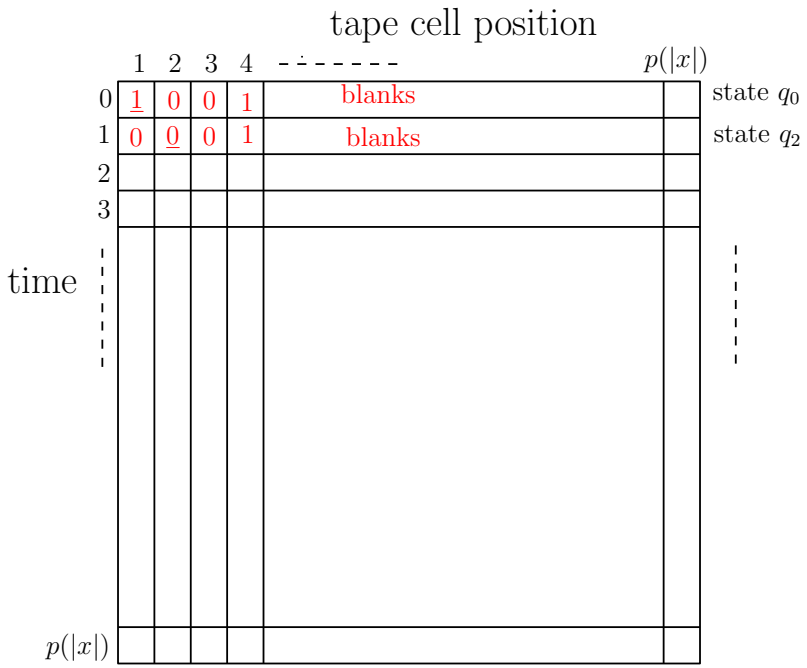
BIG IDEA

- $\mathbf{f_M(x)}$ will express "**M** on input **x** accepts in $\mathbf{p(|x|)}$ steps"
- $\mathbf{f_M(x)}$ will encode a computation history of **M** on **x**

$\mathbf{f_M(x)}$ will be a carefully constructed CNF formulat s.t if we have a satisfying assignment to it, then we will be able to see a complte accepting computation of **M** on **x** *down to the last detail* of where the head is, what transistion is chosen, what the tape contents are, at each step.

Tableu of Computation

M runs in time **p(|x|)** on **x**. Entire computation of **M** on **x** can be represented by a "tableau"



Row **i** gives contents of all cells at time **i**

At time **0** tape has input **x** followed by blanks

Each row long enough to hold all cells **M** might ever have scanned.

Variable of $f_M(x)$

Four types of variable to describe computation of M on x

- $T(b, h, i)$: tape cell at position h holds symbol b at time i.
  $1 \leq h \leq p(|x|)$, $b \in \Gamma$, $0 \leq i \leq p(|x|)$
- $H(h, i)$: read/write head is at position h at time i.
  $1 \leq h \leq p(|x|)$, $0 \leq i \leq p(|x|)$
- $S(q, i)$ state of M is q at time i $q \in Q$, $0 \leq i \leq p(|x|)$
- $I(j, i)$ instruction number j is executed at time i
  M is non-deterministic, need to specify transitions in some way.
  Number transitions as $1, 2, \ldots, \ell$ where j'th transition is
  $< q_j, b_j, q'_j, b'_j, d_j >$ indication $(q'_j, b'_j, d_j) \in \delta(q_j, b_j)$,
  direction $d_j \in \{-1, 0, 1\}$.

Number of variables is $O(p(|x|)^2)$ where constant in $O()$ hides dependence on fixed machine M.

Notation

Some abbreviations for ease of notation

$\bigwedge_{k=1}^{m} x_k$ means $x_1 \wedge x_2 \wedge \ldots \wedge x_m$

$\bigvee_{k=1}^{m} x_k$ means $x_1 \vee x_2 \vee \ldots \vee x_m$

$\bigoplus(x_1, x_2, \ldots, x_k)$ is a formula that means exactly one of $x_1, x_2, \ldots, x_m$ is true. Can be converted to CNF form

Clauses of $f_M(x)$

$f_M(x)$ is the conjunction of **8** clause groups:

$$f_M(x) = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \varphi_5 \wedge \varphi_6 \wedge \varphi_7 \wedge \varphi_8$$

where each φ_i is a CNF formula. Described in subsequent slides.
Property: $f_M(x)$ is satisfied iff there is a truth assignment to the variables that simultaneously satisfy $\varphi_1, \ldots, \varphi_8$.

φ_1

φ_1 asserts (is true iff) the variables are set T/F indicating that $\mathbf{M}$ starts in state $\mathbf{q_0}$ at time $\mathbf{0}$ with tape contents containing $\mathbf{x}$ followed by blanks.

Let $\mathbf{x = a_1 a_2 \ldots a_n}$

$\varphi_1 = \mathbf{S(q, 0)}$ state at time $\mathbf{0}$ is $\mathbf{q_0}$

$\bigwedge$ and

$\bigwedge_{\mathbf{h=1}}^{\mathbf{n}} \mathbf{T(a_h, h, 0)}$ at time $\mathbf{0}$ cells $\mathbf{1}$ to $\mathbf{n}$ have $\mathbf{a_1}$ to $\mathbf{a_n}$

$\bigwedge_{\mathbf{h=n+1}}^{\mathbf{p(|x|}})\mathbf{T(B, h, 0)}$ at time $\mathbf{0}$ cells $\mathbf{n+1}$ to $\mathbf{p(|x|)}$ have blanks

$\bigwedge$ and

$\mathbf{H(1, 0)}$ head at time $\mathbf{0}$ is in position $\mathbf{1}$

φ_2

φ_2 asserts $\mathbf{M}$ in exactly one state at any time $\mathbf{i}$

$$\varphi_2 = \bigwedge_{\mathbf{i}=\mathbf{0}}^{\mathbf{p}(|\mathbf{x}|)} \left(\bigoplus (\mathbf{S}(\mathbf{q_0}, \mathbf{i}), \mathbf{S}(\mathbf{q_1}, \mathbf{i}), \ldots, \mathbf{S}(\mathbf{q_{|Q|}}, \mathbf{i})) \right)$$

φ_3

φ_3 asserts that each tape cell holds a unique symbol at any given time.

$$\varphi_3 = \bigwedge_{i=0}^{p(|x|)} \bigwedge_{h=1}^{p(|x|)} \oplus(T(b_1, h, i), T(b_2, h, i), \ldots, T(b_{|\Gamma|}, h, i))$$

For each time i and for each cell position h exactly one symbol $b \in \Gamma$ at cell position h at time i

φ_4

φ_4 asserts that the read/write head of $\mathbf{M}$ is in exactly one position at any time $\mathbf{i}$

$$\varphi_4 = \bigwedge_{\mathbf{i}=0}^{\mathbf{p}(|\mathbf{x}|)} (\oplus (\mathbf{H}(1,\mathbf{i}), \mathbf{H}(2,\mathbf{i}), \ldots, \mathbf{H}(\mathbf{p}(|\mathbf{x}|),\mathbf{i})))$$

φ_5

φ_5 asserts that M accepts

- Let q_a be unique accept state of M
- without loss of generality assume M runs all $p(|x|)$ steps

$$\varphi_5 = S(q_a, p(|x|))$$

State at time $p(|x|)$ is q_a the accept state.

If we don't want to make assumption of running for all steps

$$\varphi_5 = \bigvee_{i=1}^{p(|x|)} S(q_a, i)$$

which means M enters accepts state at some time.

φ_6

φ_6 asserts that $\mathbf{M}$ executes a unique instruction at each time

$$\varphi_6 = \bigwedge_{i=0}^{p(|x|)} \oplus(I(1,i), I(2,i), \ldots, I(m,i))$$

where $\mathbf{m}$ is max instruction number.

φ_7

φ_7 ensures that variables don't allow tape to change from one moment to next if the read/write head was not there.

"If head is **not** at position **h** at time **i** then at time $\mathbf{i+1}$ the symbol at cell **h** must be unchanged"

$$\varphi_7 = \bigwedge_{\mathbf{i}} \bigwedge_{\mathbf{h}} \bigwedge_{\mathbf{b} \neq \mathbf{c}} \left(\overline{\mathbf{H(h,i)}} \Rightarrow \overline{\mathbf{T(b,h,i)} \bigwedge \mathbf{T(c,h,i+1)}} \right)$$

since $\mathbf{A} \Rightarrow \mathbf{B}$ is same as $\neg\mathbf{A} \vee \mathbf{B}$, rewrite above in CNF form

$$\varphi_7 = \bigwedge_{\mathbf{i}} \bigwedge_{\mathbf{h}} \bigwedge_{\mathbf{b} \neq \mathbf{c}} (\mathbf{H(h,i)} \vee \neg\mathbf{T(b,h,i)} \vee \neg\mathbf{T(c,h,i+1)})$$

φ_8

φ_8 asserts that changes in tableu/tape correspond to transitions of $\mathbf{M}$ (as Lenny says, this is the big cookie).

Let $\mathbf{j}$'th instruction be $< \mathbf{q_j, b_j, q'_j, b'_j, d_j} >$

$\varphi_8 = \bigwedge_i \bigwedge_j (\mathbf{I(j,i)} \Rightarrow \mathbf{S(q_j, i)})$ If instr $\mathbf{j}$ executed at time $\mathbf{i}$ then state must be correct to do $\mathbf{j}$

$\bigwedge$

$\bigwedge_i \bigwedge_j (\mathbf{I(j,i)} \Rightarrow \mathbf{S(q'_j, i+1)})$ and at next time unit, state must be the proper next state for instr $\mathbf{j}$

$\bigwedge$

$\bigwedge_i \bigwedge_h \bigwedge_j [(\mathbf{I(j,i)} \bigwedge \mathbf{H(h,i)}) \Rightarrow \mathbf{T(b_j, h, i)}]$ if $\mathbf{j}$ was executed and head was at position $\mathbf{h}$, then cell $\mathbf{h}$ has correct symbol for $\mathbf{j}$ $\bigwedge$

$\bigwedge_i \bigwedge_j \bigwedge_h [(\mathbf{I(j,i)} \wedge \mathbf{H(h,i)}) \Rightarrow \mathbf{T(b'_j, h, i+1)}]$ if $\mathbf{j}$ was done then at time $\mathbf{i}$ with head at $\mathbf{h}$ then at next time step symbol $\mathbf{b'_j}$ was indeed written in position $\mathbf{h}$ $\bigwedge$

$\bigwedge_i \bigwedge_j \bigwedge_h [(\mathbf{I(j,i)} \wedge \mathbf{H(h,i)}) \Rightarrow \mathbf{H(h+d_j, i+1)}]$ and head is moved properly according to instr $\mathbf{j}$.

Proof of Correctness

(Sketch)

- Given M, x, poly-time algorithm to construct $f_M(x)$
- if $f_M(x)$ is satisfiable then the truth assignment completely specifies an accepting computation of M on x
- if M accepts x then the accepting computation leads to an "obvious" truth assignment to $f_M(x)$. Simply assign the variables according to the state of M and cells at each time i.

Thus M accepts x if and only if $f_M(x)$ is satisfiable