

1. Suppose you are given a sequence of non-negative integers separated by + and  $\times$  signs; for example:

$$2 \times 3 + 0 \times 6 \times 1 + 4 \times 2$$

You can change the value of this expression by adding parentheses in different places. For example:

$$2 \times (3 + (0 \times (6 \times (1 + (4 \times 2)))))) = 6$$

$$((((2 \times 3) + 0) \times 6) \times 1) + 4) \times 2 = 80$$

$$((2 \times 3) + (0 \times 6)) \times (1 + (4 \times 2)) = 108$$

$$(((2 \times 3) + 0) \times 6) \times ((1 + 4) \times 2) = 360$$

Describe and analyze an algorithm to compute, given a list of integers separated by + and  $\times$  signs, the largest possible value we can obtain by inserting parentheses.

Your input is an array  $A[0..2n]$  where each  $A[i]$  is an integer if  $i$  is even and + or  $\times$  if  $i$  is odd. Assume any arithmetic operation in your algorithm takes  $O(1)$  time.

## Basic steps in developing a dynamic programming algorithm

1. **Formulate the problem recursively.** This is the hard part. There are two distinct but equally important things to include in your formulation.
  - (a) **Specification.** First, give a clear and precise English description of the problem you are claiming to solve. Don't describe *how* to solve the problem at this stage; just describe *what* the problem actually is. Otherwise, the reader has no way to know what your recursive algorithm is *supposed* to compute.
  - (b) **Solution.** Second, give a clear recursive formula or algorithm for the whole problem in terms of the answers to smaller instances of *exactly* the same problem. It generally helps to think in terms of a recursive definition of your inputs and outputs. If you discover that you need a solution to a *similar* problem, or a slightly *related* problem, you're attacking the wrong problem; go back to step 1.
  
2. **Build solutions to your recurrence from the bottom up.** Write an algorithm that starts with the base cases of your recurrence and works its way up to the final solution, by considering intermediate subproblems in the correct order. This stage can be broken down into several smaller, relatively mechanical steps:
  - (a) **Identify the subproblems.** What are all the different ways can your recursive algorithm call itself, starting with some initial input? For example, the argument to RECFIBO is always an integer between 0 and  $n$ .
  - (b) **Analyze space and running time.** The number of possible distinct subproblems determines the space complexity of your memoized algorithm. To compute the time complexity, add up the running times of all possible subproblems, *ignoring the recursive calls*. For example, if we already know  $F_{i-1}$  and  $F_{i-2}$ , we can compute  $F_i$  in  $O(1)$  time, so computing the first  $n$  Fibonacci numbers takes  $O(n)$  time.
  - (c) **Choose a data structure to memoize intermediate results.** For most problems, each recursive subproblem can be identified by a few integers, so you can use a multidimensional array. For some problems, however, a more complicated data structure is required.
  - (d) **Identify dependencies between subproblems.** Except for the base cases, every recursive subproblem depends on other subproblems—which ones? Draw a picture of your data structure, pick a generic element, and draw arrows from each of the other elements it depends on. Then formalize your picture.
  - (e) **Find a good evaluation order.** Order the subproblems so that each subproblem comes *after* the subproblems it depends on. Typically, this means you should consider the base cases first, then the subproblems that depends only on base cases, and so on. More formally, the dependencies you identified in the previous step define a partial order over the subproblems; in this step, you need to find a linear extension of that partial order. ***Be careful!***
  - (f) **Write down the algorithm.** You know what order to consider the subproblems, and you know how to solve each subproblem. So do that! If your data structure is an array, this usually means writing a few nested for-loops around your original recurrence.