Binomials, Derangements, and Catalan Numbers

Dr. Mattox Beckman

University of Illinois at Urbana-Champaign
Department of Computer Science

Objectives

Your Objectives:

- Know how to calculate and use these sequences:
  - Fibonacci
  - Binomial Coefficients
  - Derangements
  - Catalan Numbers

Zeckenorf's Theorem

- We've talked about Fibonacci numbers already, but one tidbit:
- Every positive integer $n = f_i + f_j$ where $i + 1 > j$. Try proving it!
- Use a greedy algorithm to find f_i and f_j.

Binomial Coefficients

- Coefficients of the expansion of $(x + y)^n$
  e.g. $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
- Number of ways to chose k items from n objects. (k starts at 0...)
- The formula: $C(n, k) = \frac{n!}{k!(n-k)!}$
- The recurrence: "either take or ignore an item"
  $$\begin{aligned} C(n,0) = C(n,n) = &\quad 1 \\ C(n,k) = &\quad C(n-1,k-1) + C(n-1,k) \end{aligned}$$
- Use DP if you need a lot, but not all, of these numbers.

Derangements

- Number of permutations of n is $n!$.
- But... how many ways are there to make a permutation such that no element is in its original spot?
- Written $!n$

$$\begin{aligned} !0 &= \quad 0 \\ !1 &= \quad 0 \\ !n &= \quad (n-1) * (!(n-1) + !(n-2)) \end{aligned}$$

- $!2 = 1, !3 = 2, !4 = 9, !5 = 44, !6 = 265, \ldots$
- Not that common, but easy to code with DP.

Catalan Numbers

- This sequence has a *lot* of isomorphisms.
- $Cat(n) = C(2 \times n, n)/(n+1); Cat(0) = 1$
- Recursively: $Can(n+1) = \frac{(2n+2)(2n+1)}{(n+2)(n+1)} Cat(n)$
- $Cat(0) = 1, Cat(1) = 1, Cat(2) = 2, Cat(3) = 5, Cat(4) = 14, \ldots$
- Some things Catalan numbers count:
  - $Cat(n)$ – Number of distinct binary trees with n vertices.
  - Number of ways $n+1$ factors can be completely parethesized: abcd = a(b(cd)) = ((ab)c)d = (ab)(cd) = a((bc)d) = (a(bc))d
  - Number of ways a convex polygon can be triangulated.