HW 3 – Interpretation and Proof of First Order Logic Formulae

CS 477 – Spring 2020
Revision 1.0

Assigned February 26, 2020
Due March 4, 2020, 9:00 pm
Extension 48 hours (20% penalty)

1 Change Log

1.0 Initial Release.

2 Objectives and Background

The purpose of this HW is to test your understanding of

- modeling and interpretation of first order logic formulae

Another purpose of HWs is to provide you with experience answering non-programming written questions of the kind you may experience on the midterm and final.

3 Turn-In Procedure

The pdf for this assignment (`hw3.pdf`) should be found in the `assignments/hw3/` subdirectory of your git directory for this course. Your solution should be put in that same directory. Using your favorite tool(s), you should put your solution in a file named `hw3-submission.pdf`. If you have problems generating a pdf, please seek help from the course staff. Your answers to the following questions are to be submitted electronically from within `assignments/hw3/` subdirectory by committing the file as follows:

```
git add hw3-submission.pdf
git commit -m "Turning in hw3"
git push
```

4 Problems

Each of the formulae in Problems 1-3 is over the signature

$$\mathcal{G} = (V = \{u, v, w, x, y, z\}, F = \{+\}, af = \{+ \mapsto 2;\}, R = \{=, <\}, ar = \{= \mapsto 2, < \mapsto 2\})$$

. The operator $+$ and the relations $=$ and $<$ will be written as infixed. For each of the formulae in Problems 1-3, give the following:

a. (2 pts) the list of free variables;

b. for the structure $\mathcal{S} = \{\mathcal{G}, \mathcal{D} = \mathbb{N}, \mathcal{F}, \phi, \mathcal{R}, \rho\}$ where $\phi(+)$ is normal addition, and where $\rho(=)$ is normal equality and $\rho(<)$ is normal less-than comparison. ($\mathbb{N}$ is the non-negative integers.)

(i) (3 pts) give an assignment for which the formula is valid and say why the assignment satisfies the formula, or say why none exists, and

(ii) (3 pts) give an assignment for which the formula is invalid and say why the assignment fails to satisfy the formula, or say why none is possible;

c. for the structure $\mathcal{S} = \{\mathcal{G}, \mathcal{D} = \mathbb{R}, \mathcal{F}, \phi, \mathcal{R}, \rho\}$ where $\phi(+)$ is multiplication, and where $\rho(=)$ is normal equality but where $\rho(<)(x, y) = (x^2 < y^2)$.

(i) (3 pts) give an assignment for which the formula is valid and say why the assignment satisfies the formula, or say why none exists, and

(ii) (3 pts) give an assignment for which the formula is invalid and say why the assignment fails to satisfy the formula, or say why none is possible;

1. $\exists u.\ \forall v.\ (u < x) \wedge ((u < v) \Rightarrow ((x = v) \vee (x < v)))$

2. $(u < v) \wedge (v < w) \Rightarrow ((\exists x.\ ((u < x) \wedge (x < v))) \wedge (\exists u.(v < u) \wedge (u < w)))$

3. $\forall x.\ \forall y.(((x < y) \vee (x = y)) \wedge ((y < x) \vee (x = y))) \Rightarrow (x = y)$

For Problems 4 and 5, give a proof in the sequent encoding of Natural Deduction for First Order Logic of the given formulae. You may give your proofs as written proof trees, or you may do your proofs in Isabelle and submit a separate `hw3.thy` file containing the proofs. The same restictions as for MP1 apply, except that you are also allowed to use the introduction and elimination rules `allI`, `allE`, `exI` and `exE`.

4. (10pts) $(\forall x.\forall y.P(x) \Rightarrow Q(y)) \Rightarrow ((\exists x.P(x)) \Rightarrow (\forall y.Q(y)))$

5. (12 pts) $(\forall x.\forall y.P(x) \wedge P(y)) \Rightarrow ((\forall x.P(x)) \wedge (\forall y.P(y)))$

5 Extra Credit

6. (5 pts) Give a structure that models the formula in Problem 1 (different for either structure I gave) and describe why the structure models the formula.