CS477 Formal Software Dev Methods

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Slides based in part on previous lectures
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Proof implies Truth

Theorem (Soundness)

Suppose $\{H_1, \ldots, H_n\} \vdash P$ *is provable. Then, for every valuation* v*, if for every* i *we have* $v \models H_i$*, then* $v \models P$*.*

Proof.

- Fix a proof of $\{H_1, \ldots, H_n\} \vdash P$

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- Fix a proof of $\{H_1, \ldots, H_n\} \vdash P$
- Proceed by induction on the structure of the proof tree of $\{H_1, \ldots, H_n\} \vdash P$.

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- Fix a proof of $\{H_1, \ldots, H_n\} \vdash P$
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- Ind Hyp: We may assume that, for every subproof of the proof of $\{H_1, \ldots, H_n\} \vdash P$, if v satisfies all the hypotheses of the result of the subproof, then v satisfies the consequent of the result of the subproof.

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- Case: Hyp
  - The P is among the H_i, so by assumption $v \models P$.

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Proof.

- Case: **T I**

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Proof.

- Case: $\mathbf{T}$ I
  - Then $P = \mathbf{T}$ and $v \models \mathbf{T}$ always.

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- Case: And I

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  - Then $P = \mathbf{T}$ and $v \models \mathbf{T}$ always.
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  - Then there exist A and B s.t. $P = A \wedge B$ and $\{H_1, \ldots, H_n\} \vdash A$ and $\{H_1, \ldots, H_n\} \vdash B$ are provable by subproofs of the proof of $\{H_1, \ldots, H_n\} \vdash P$.

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  - By inductive hypothesis, since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A$ and $v \models B$.
  - Thus $v \models A \wedge B$ so $v \models P$.

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  - By inductive hypothesis, since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A$.

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  - By inductive hypothesis, since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A$.
  - Thus $v \models A \vee B$ so $v \models P$.
- Case Or_R **I** same.

Proof implies Truth

Proof.

- Case: Not I

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    - Then there exists A s.t. $P = \neg A$ and $\{H_1, \ldots, H_n, A\} \vdash \mathbf{F}$ is provable by a subproof of the proof of $\{H_1, \ldots, H_n\} \vdash P$.

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  - Have $v \models H_i$ for $i = 1 \ldots n$, but not $v \models \mathbf{F}$.

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  - Have $v \models H_i$ for $i = 1 \ldots n$, but not $v \models \mathbf{F}$.
  - By Ind. Hyp. must have $v \not\models A$

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- Case: Imp I

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  - By Ind. Hyp. must have $v \not\models A$
  - Thus $v \models \neg A$.
- Case: Imp I
  - Then there exist A and B s.t. $P = A \Rightarrow B$ and $\{H_1, \ldots, H_n, A\} \vdash B$ is provable by a subproof of the proof of $\{H_1, \ldots, H_n\} \vdash P$.

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  - By inductive hypothesis, since $v \models H_i$ for $i = 1 \ldots n$, if $v \models A$ then $v \models B$, so either have $v \models B$ or $v \not\models A$.

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  - Thus $v \models A \Rightarrow B$ so $v \models P$.

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Proof.

- Case Not E

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- Case Not E
  - Then there exist A s.t. $\{H_1, \ldots, H_n\} \vdash \neg A$ and $\{H_1, \ldots, H_n\} \vdash A$ are provable by subproofs of $\{H_1, \ldots, H_n\} \vdash P$.

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  - By inductive hypothesis, since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A$ and $v \models \neg A$, which is imposible.

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  - Thus either the last rule is not Not E or for some i we have $v \not\models H_i$, contradicting theorem assumption.

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  - Then there exist A and B s.t. $\{H_1, \ldots, H_n\} \vdash A \Rightarrow B$ and $\{H_1, \ldots, H_n\} \vdash A$ and $\{H_1, \ldots, H_n, B\} \vdash P$ are provable by subproofs of $\{H_1, \ldots, H_n\} \vdash P$.

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  - By Ind. Hyp., since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A \Rightarrow B$ and $v \models A$.

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  - By Ind. Hyp., since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A \Rightarrow B$ and $v \models A$.
  - Therefore $v \models B$.

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  - By inductive hypothesis, since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A$ and $v \models \neg A$, which is imposible.
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- Case: Imp E
  - Then there exist A and B s.t. $\{H_1, \ldots, H_n\} \vdash A \Rightarrow B$ and $\{H_1, \ldots, H_n\} \vdash A$ and $\{H_1, \ldots, H_n, B\} \vdash P$ are provable by subproofs of $\{H_1, \ldots, H_n\} \vdash P$.
  - By Ind. Hyp., since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A \Rightarrow B$ and $v \models A$.
  - Therefore $v \models B$.
  - Again by Ind. Hyp, $v \models P$.

Proof implies Truth

Proof.

- Case: And_L E

□

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Proof.

- Case: And$_L$ E
    - Then there exist A and B s.t. $\{H_1, \ldots, H_n\} \vdash A \wedge B$ and $\{H_1, \ldots, H_n, A\} \vdash P$ are provable by subproofs of $\{H_1, \ldots, H_n\} \vdash P$.

□

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Proof.

- Case: And$_L$ E
  - Then there exist A and B s.t. $\{H_1, \ldots, H_n\} \vdash A \land B$ and $\{H_1, \ldots, H_n, A\} \vdash P$ are provable by subproofs of $\{H_1, \ldots, H_n\} \vdash P$.
  - By Ind. Hyp., since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A \land B$, so $v \models A$ (and $v \models B$).

□

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- Case: And$_L$ E
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  - By Ind. Hyp., since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A \wedge B$, so $v \models A$ (and $v \models B$).
  - Again by Ind. Hyp, $v \models P$.

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- Case: And_L E
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  - By Ind. Hyp., since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A \wedge B$, so $v \models A$ (and $v \models B$).
  - Again by Ind. Hyp, $v \models P$.
- Case: And_R E same.

□

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- Case: And$_L$ E
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- Case: And$_R$ E same.
- Case: **F** E

□

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- Case: And_L E
  - Then there exist A and B s.t. $\{H_1, \ldots, H_n\} \vdash A \wedge B$ and $\{H_1, \ldots, H_n, A\} \vdash P$ are provable by subproofs of $\{H_1, \ldots, H_n\} \vdash P$.
  - By Ind. Hyp., since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A \wedge B$, so $v \models A$ (and $v \models B$).
  - Again by Ind. Hyp, $v \models P$.
- Case: And_R E same.
- Case: **F** E
  - Then $\{H_1, \ldots, H_n\} \vdash \mathbf{F}$ is provable by a subproof of $\{H_1, \ldots, H_n\} \vdash P$.

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- Case: And_L E
  - Then there exist A and B s.t. $\{H_1, \ldots, H_n\} \vdash A \wedge B$ and $\{H_1, \ldots, H_n, A\} \vdash P$ are provable by subproofs of $\{H_1, \ldots, H_n\} \vdash P$.
  - By Ind. Hyp., since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A \wedge B$, so $v \models A$ (and $v \models B$).
  - Again by Ind. Hyp, $v \models P$.
- Case: And_R E same.
- Case: $\mathbf{F}$ E
  - Then $\{H_1, \ldots, H_n\} \vdash \mathbf{F}$ is provable by a subproof of $\{H_1, \ldots, H_n\} \vdash P$.
  - By Ind. Hyp., since $v \models H_i$ for $i = 1 \ldots n$, have $v \models \mathbf{F}$, which is impossible.

□

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- Case: And$_L$ E
  - Then there exist A and B s.t. $\{H_1, \ldots, H_n\} \vdash A \wedge B$ and $\{H_1, \ldots, H_n, A\} \vdash P$ are provable by subproofs of $\{H_1, \ldots, H_n\} \vdash P$.
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- Case: **F** E
  - Then $\{H_1, \ldots, H_n\} \vdash \mathbf{F}$ is provable by a subproof of $\{H_1, \ldots, H_n\} \vdash P$.
  - By Ind. Hyp., since $v \models H_i$ for $i = 1 \ldots n$, have $v \models \mathbf{F}$, which is impossible.
  - Therefore, either last rule in proof not **F** E, or $v \neg \models H_i$, which violates theorem assumption.

□

Proof implies Truth

Proof.

- Case: Or E
  - Then there exist A and B s.t. $\{H_1, \ldots, H_n\} \vdash A \vee B$ and $\{H_1, \ldots, H_n, A\} \vdash P$ and $\{H_1, \ldots, H_n, B\} \vdash P$ are all provable by subproofs of $\{H_1, \ldots, H_n\} \vdash P$.

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  - By Ind. Hyp., since $v \models H_i$ for $i = 1 \ldots n$, have $v \models A \vee B$.
  - Have at least one of $v \models A$ or $v \models B$.

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  - Have at least one of $v \models A$ or $v \models B$.
  - Case: $v \models A$
    - Ind. Hyp. implies $v \models P$.
  - Case: $v \models B$
    - Again by Ind. Hyp, $v \models P$.

□

Truth does not imply Proof . . .

- For given rules, can not prove $A \vee \neg A$
- Need an axiom.

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- BDDs are one such

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  - Left edges labeled false and right edges labeled true.
    - Think 0 and 1
  - For each path (branch) in the tree, each atomic proposition may label at most one vertex of that path.

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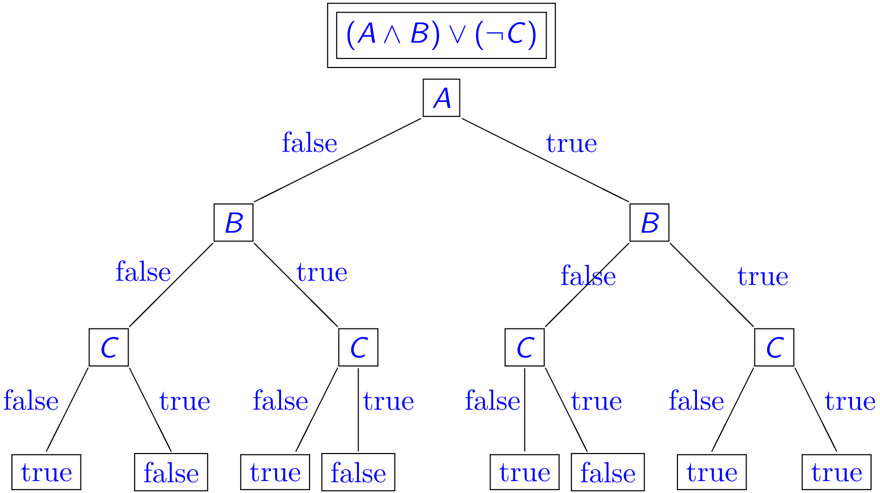
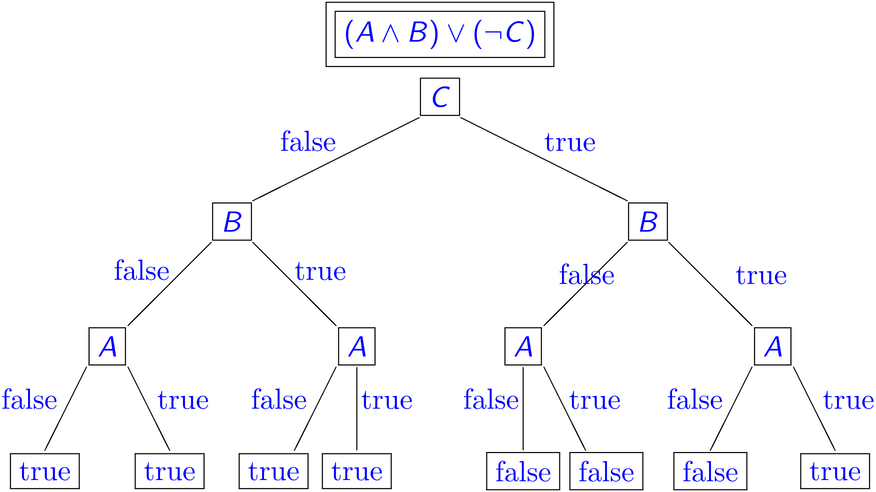
    - Is a model (true, the tree accepts the valuation)
    - Or not a model (false, the tree rejects the valuation)
  - Each valuation matches exactly one branch
  - More than one valuation may (will) match a given branch

Example:



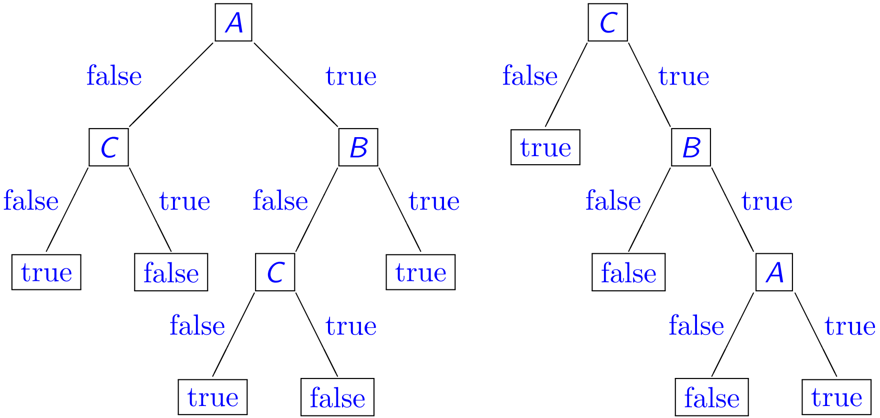
$(A \wedge B) \vee (\neg C)$
A
false
true
B
B
false
true
false
true
C
C
C
C
false
true
false
true
false
true
false
true
true
false
true
false
true
false
true
true

Example: Different Variable Ordering - Different Tree



Example: Many Logically Equivalent Trees

$(A \wedge B) \vee (\neg C)$



Alternate Syntax for Propositional Logic

- Still have constants $\{\mathbf{T}, \mathbf{F}\}$
- Still have countable set AP of propositional variables a.k.a. atomic propositions
- Only one ternary connective: the conditional if_then_else_
  - First argument only a variable
  - Second and third arguments propositions
  - Example

$$\textit{if } C \textit{ then if } B \textit{ then if } A \textit{ then } \mathbf{T} \textit{ else } \mathbf{F} \textit{ else } \mathbf{F} \textit{ else } \mathbf{T}$$

  - Represents the last tree above

Semantics for Conditional Propositional Logic

- Define when a valuation v satisfies a conditional proposition by

$$
\begin{array}{l}
v \models \mathbf{T} \\
v \not\models \mathbf{F} \\
v \models \textit{if } A \textit{ then } P_t \textit{ else } P_f \text{ iff} \\
\qquad v(A) = \mathrm{true} \text{ and } v \models P_t \text{ or} \\
\qquad v(A) = \mathrm{false} \text{ and } v \models P_f
\end{array}
$$

- Example: let $v = \{A \mapsto \mathrm{true},\ B \mapsto \mathrm{true}, C \mapsto \mathrm{true}\}$

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  $v(C) = \mathrm{true}$ and
  $v \models \mathit{if}\ B\ \mathit{then}\ \mathit{if}\ A\ \mathit{then}\ \mathbf{T}\ \mathit{else}\ \mathbf{F}\ \mathit{else}\ \mathbf{F}$

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$v(C) = \mathrm{true}$ and
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$v(A) = \mathrm{true}$ and$v \models \mathbf{T}$

Translating Original Propositions into if_then_else

- Start with proposition P_0 with variables $v_1, \ldots v_n$
- $P[c/v]$ is the proposition resulting from replacing all occurrences of variable v with constant c
- Let $\overline{P}$ be the result of evaluating every subexpression of P containing no variables
- Let $P_1 = \mathit{if}\ \ v_1\ \ \mathit{then}\ \overline{P_0[\mathbf{T}/v_1]}\ \mathit{else}\ \overline{P_0[\mathbf{F}/v_1]}$
- Let $P_i = \mathit{if}\ \ v_i\ \ \mathit{then}\ P_{i-1}[\mathbf{T}/v_i]\ \mathit{else}\ P_{i-1}[\mathbf{F}/v_i]$
- P_n is logically equivalent to P, but only uses if_then_else_.
  - Valuation satisfies P if and only if it satisfies P_n
  - P_n depends on the order of variables $v_1, \ldots v_n$
  - P_n directly corresponds to a binary decision tree

Example:

$P = (A \land B) \lor (\neg C)$, variables $\{A,\ B,\ C\}$, $A < B < C$

$P_0 = (A \land B) \lor (\neg C)$

Example:

$P = (A \wedge B) \vee (\neg C)$, variables $\{A,\ B,\ C\}$, $A < B < C$

$$P_0 = (A \wedge B) \vee (\neg C)$$
$$P_1 = \mathit{if}\ A\ \mathit{then}\ (\mathbf{T}\ \wedge\ B) \vee (\neg C)\ \mathit{else}\ (\mathbf{F}\ \wedge\ B) \vee (\neg C)$$

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\begin{aligned}
P_0 &= (A \wedge B) \vee (\neg C) \\
P_1 &= \mathit{if}\ A\ \mathit{then}\ (\mathbf{T}\ \wedge\ B) \vee (\neg C)\ \mathit{else}\ (\mathbf{F}\ \wedge\ B) \vee (\neg C) \\
P_2' &= \mathit{if}\ B\ \mathit{then}\ (\mathit{if}\ A\ \mathit{then}\ (\mathbf{T}\ \wedge\ \mathbf{T}) \vee (\neg C)\ \mathit{else}\ (\mathbf{F}\ \wedge\ \mathbf{T}) \vee (\neg C)) \\
&\qquad\qquad \mathit{else}\ (\mathit{if}\ A\ \mathit{then}\ (\mathbf{T}\ \wedge\ \mathbf{F}) \vee (\neg C)\ \mathit{else}\ (\mathbf{F}\ \wedge\ \mathbf{F}) \vee (\neg C))
\end{aligned}
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&\qquad\qquad else\ (if\ \ A\ then\ \mathbf{F} \vee (\neg C)\ else\ \mathbf{F} \vee (\neg C))\\
P_3' &= if\ \ C\ then\ (if\ \ B\ then\ (if\ \ A\ then\ \mathbf{T} \vee (\neg \mathbf{T})\ else\ \mathbf{F} \vee (\neg \mathbf{T}))\\
&\qquad\qquad\qquad\qquad else\ (if\ \ A\ then\ \mathbf{F} \vee (\neg \mathbf{T})\ else\ \mathbf{F} \vee (\neg \mathbf{T})))\\
&\qquad\qquad else\ (if\ \ B\ then\ (if\ \ A\ then\ \mathbf{T} \vee (\neg \mathbf{F})\ else\ \mathbf{F} \vee (\neg \mathbf{F}))\\
&\qquad\qquad\qquad\qquad else\ (if\ \ A\ then\ \mathbf{F} \vee (\neg \mathbf{F})\ else\ \mathbf{F} \vee (\neg \mathbf{F})))
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P_3 &= \mathit{if}\ C\ \mathit{then}\ (\mathit{if}\ B\ \mathit{then}\ (\mathit{if}\ A\ \mathit{then}\ \mathbf{T}\ \mathit{else}\ \mathbf{F})\\
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&\qquad \mathit{else}\ (\mathit{if}\ B\ \mathit{then}\ (\mathit{if}\ A\ \mathit{then}\ \mathbf{T}\ \mathit{else}\ \mathbf{T})\\
&\qquad\qquad \mathit{else}\ (\mathit{if}\ A\ \mathit{then}\ \mathbf{T}\ \mathit{else}\ \mathbf{T}))
\end{aligned}
$$

Example, cont.

$$\begin{aligned} P_3 = \mathit{if}\ C\ \mathit{then}\ (&\mathit{if}\ B\ \mathit{then}\ (\mathit{if}\ A\ \mathit{then}\ \mathbf{T}\ \mathit{else}\ \mathbf{F}) \\ &\qquad \mathit{else}\ (\mathit{if}\ A\ \mathit{then}\ \mathbf{F}\ \mathit{else}\ \mathbf{F})) \\ &\mathit{else}\ (\mathit{if}\ B\ \mathit{then}\ (\mathit{if}\ A\ \mathit{then}\ \mathbf{T}\ \mathit{else}\ \mathbf{T}) \\ &\qquad \mathit{else}\ (\mathit{if}\ A\ \mathit{then}\ \mathbf{T}\ \mathit{else}\ \mathbf{T})) \end{aligned}$$

P_3 corresponds to second binary decision tree given earlier

- Any proposition in strict if_then_else_ form corresponds directly to a binary decision tree that accepts exactly the valuations that satisfy (model) the proposition.

Binary Decision Diagram

- Binary decision trees may contain (much) redundancy
- Binary Decision Diagram (BDD): Replace trees by (rooted) directed acyclic graphs
- Require all other conditions still hold
- Generalization of binary decision trees
- Allows for sharing of common subtrees.
- Accepts / rejects valuations as with binary decision trees.

Example

Reduced Ordered Binary Decision Diagrams

- Problem: given proposition may correspond to many different BDDs
- How to create a (compact) canonical BDD for a proposition such that two different propositions are logically equivalent if and only if they have the same (isomorphic) canonical BDD
- Start: order propositional variables $v_i < v_j$.
- Bryant showed you can obtain such a canonical BDD by requiring
  - Variables should appear in order on each path for root to leaf
  - No distinct duplicate (isomorphic) subtrees (including leaves)

Achieving Canonical Form

- Start with an Ordered BDD (all edges in correct order)
- Repeat following until none apply
- Remove duplicate leaves: Eliminate all but one leaf with a given label and redirect all edges to the eliminated leaves to the remaining one
- Remove duplicate nonterminals: If node n and m have the same variable label, their left edges point to the same node and their right edges point to the same node, remove one and redirect edges that pointed to it to the other
- Remove redundant tests: If both out edges of node n point to node m, eliminate n and redirect all edges coming into n to m
- Bryant gave procedure to do the above that terminates in linear time

Example