

CS477 Formal Software Dev Methods

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Slides based in part on previous lectures
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Example Tautology

$$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$$

A	B	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow B$	$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$
true	false			

$$v = \{A \mapsto \text{True}; B \mapsto \text{False}\}$$

$$\mathcal{I}_{\{A \mapsto \text{true}; B \mapsto \text{false}\}}(A) = \text{true}$$

$$\mathcal{I}_{\{A \mapsto \text{true}; B \mapsto \text{false}\}}(B) = \text{false}$$

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Non-standard Model of Propositional Logic

Other models possible

Example:

- $\mathcal{C} = \{\text{true}, \text{false}, \perp\}$
- Valuations assign values in \mathcal{C} to propositional atoms
- If $\mathcal{J}_w(p) = \perp$ then $\mathcal{J}_w(\neg p) = \perp$, otherwise same as for \mathcal{I}
- $\mathcal{J}_w(p) = \perp$ or $\mathcal{J}_w(q) = \perp$ then $\mathcal{J}_w(\neg p) = \perp$, $\mathcal{J}_w(p \wedge q) = \perp$, $\mathcal{J}_w(p \vee q) = \perp$, $\mathcal{J}_w(p \Rightarrow q) = \perp$, and $\mathcal{J}_w(p \Leftrightarrow q) = \perp$; otherwise same as for \mathcal{I}
- Note: $A \vee \neg A \neq \mathbf{T}$

- Other variants possible

Proofs in Propositional Logic

- Natural Deduction proof is tree and a **discharge function**
 - Nodes are instances of inference rules
 - Leaves are assumptions of subproofs
 - Discharge function maps each leaf of the tree to an ancestor as allowed by the inference rules

Natural Deduction Inference Rules

- Inference rule has hypotheses and conclusion
- Conclusion a single proposition
- Hypotheses zero or more propositions, possibly with (discharged) hypotheses
- Rule with no hypotheses called an **axiom**
- Inference rule graphically presents as

$$\frac{H_1 \quad \dots \quad \begin{array}{c} A_i \\ \vdots \\ H_i \end{array} \quad \dots \quad H_j \quad \dots \quad \begin{array}{c} A_k \\ \vdots \\ H_k \end{array} \quad \dots \quad H_n}{C} \text{ rule}$$

Natural Deduction Inference Rules

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- Two main kinds of inference rules:
 - Introduction – says how to conclude proposition made from connective is true

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- Eliminations – says how to use a proposition made from connective to prove result
 - Example:

$$\frac{A \Rightarrow B \quad A \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C} \text{ Imp E}$$

Introduction Rules

Truth Introduction:

$$\frac{}{\mathbf{T}} \text{ T I}$$

And Introduction:

$$\frac{A \quad B}{A \wedge B} \text{ And I}$$

Or Introduction:

$$\frac{A}{A \vee B} \text{ Or}_L \text{ I}$$

$$\frac{B}{A \vee B} \text{ Or}_R \text{ I}$$

Not Introduction:

$$\frac{\begin{array}{c} A \\ \vdots \\ \mathbf{F} \end{array}}{\neg A} \text{ Not I}$$

Implication Introduction:

$$\frac{\begin{array}{c} A \\ \vdots \\ B \end{array}}{A \Rightarrow B} \text{ Imp I}$$

No False Introduction

Example Proof 1

$$A \Rightarrow (B \Rightarrow (A \wedge B))$$

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$$\frac{\frac{A}{\quad}}{B \Rightarrow (A \wedge B)} \quad \text{Imp I}$$
$$\frac{}{A \Rightarrow (B \Rightarrow (A \wedge B))}$$

Example Proof 1

$$\frac{\frac{A \quad B}{A \wedge B}}{B \Rightarrow (A \wedge B)} \text{Imp I}$$
$$\frac{B \Rightarrow (A \wedge B)}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{Imp I}$$

Example Proof 1

$$\frac{\frac{A \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$
$$\frac{B \Rightarrow (A \wedge B)}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{ Imp I}$$

Example Proof 1

$$\frac{\frac{A \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$
$$\frac{B \Rightarrow (A \wedge B)}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{ Imp I}$$

- All assumptions discharged; proof complete

Example Proof 2

$$\overline{B \Rightarrow (A \wedge B)}$$

Example Proof 2

$$\frac{B}{B \Rightarrow (A \wedge B)}$$

Example Proof 2

$$\frac{\frac{B}{A \wedge B}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$

Example Proof 2

$$\frac{\frac{A \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$

Example Proof 2

$$\frac{\frac{A? \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$

Example Proof 2

$$\frac{\frac{A \quad B}{A \wedge B} \text{ And I}}{B \Rightarrow (A \wedge B)} \text{ Imp I}$$

- Closed proofs must discharge all hypotheses
- Otherwise have theorem relative to / under undischarged hypotheses
- Here have proved “Assuming A , we have $B \Rightarrow (A \wedge B)$ ”

Discharging Hypothesis

$$\frac{}{A \Rightarrow (A \wedge A)}$$

Discharging Hypothesis

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- **Imp I** (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\frac{\frac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I} \qquad \frac{}{A \Rightarrow (B \Rightarrow A)}$$

- **Imp I** (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\frac{\frac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I}$$

$$\frac{\frac{A}{B \Rightarrow A} \text{ Imp I}}{A \Rightarrow (B \Rightarrow A)} \text{ Imp I}$$

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Discharging Hypothesis

$$\frac{\frac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I}$$

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Discharging Hypothesis

$$\frac{\frac{A \quad A}{A \wedge A} \text{ And I}}{A \Rightarrow (A \wedge A)} \text{ Imp I}$$

$$\frac{\frac{A}{B \Rightarrow A} \text{ Imp I}}{A \Rightarrow (B \Rightarrow A)} \text{ Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis
- Or may discharge none at all
- Every assumption instance discharged only once

Your Turn

$$A \Rightarrow (A \vee B)$$

Elimination Rules

- So far, have rules to “introduce” logical connectives into propositions
- No rules for how to “use” logical connectives
 - No assumptions with logical connectives
- Need “elimination” rules
- Example: Can’t prove

$$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$$

with what we have so far

- Elimination rules assume assumption with a connective; have general conclusion
 - Generally needs additional hypotheses

Elimination Rules

False Elimination:

$$\frac{F}{C} \text{ F E}$$

Not Elimination:

$$\frac{\neg A \quad A}{C} \text{ Not E}$$

And Elimination:

$$\frac{A \wedge B \quad \begin{array}{c} A \\ \vdots \\ C \end{array}}{C} \text{ And}_L \text{ E}$$

$$\frac{A \wedge B \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C} \text{ And}_R \text{ E}$$

Or Elimination:

$$\frac{A \vee B \quad \begin{array}{c} A \\ \vdots \\ C \end{array} \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C} \text{ Or E}$$

Implication Elimination:

$$\frac{A \Rightarrow B \quad A \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C} \text{ Imp E}$$

Example Proof 4

$$\overline{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}$$

Example Proof 4

$$\frac{\overline{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

Example Proof 4

$$\frac{\frac{\overline{A \Rightarrow C}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

Example Proof 4

$$\frac{\frac{\frac{C}{A \Rightarrow C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}$$

Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A \quad \frac{\quad C}{\quad} \text{Imp E}}{C} \text{Imp I}}{A \Rightarrow C} \text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A \quad \frac{\quad C}{\quad} \text{Imp E}}{C} \text{Imp I}}{A \Rightarrow C} \text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A \quad \frac{C}{C}}{C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I} \text{ Imp E}$$

Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A}{A} \quad \frac{\frac{B \Rightarrow C \quad B \quad C}{C} \text{ Imp E}}{C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}$$

Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A}{A} \quad \frac{\frac{B \Rightarrow C \quad B \quad C}{C} \text{ Imp E}}{C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}$$

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$$\frac{\frac{\frac{A \Rightarrow B \quad A}{C} \text{ Imp E} \quad \frac{B \Rightarrow C \quad B \quad C}{C} \text{ Imp E}}{C} \text{ Imp I}}{A \Rightarrow C} \text{ Imp I}}{\frac{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I}} \text{ Imp I}$$

Example Proof 4

$$\frac{\frac{\frac{A \Rightarrow B \quad A}{A} \quad \frac{\frac{B \Rightarrow C \quad B \quad C}{C} \text{ Imp E}}{C} \text{ Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{ Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{ Imp I} \text{ Imp E}$$

Some Well-Known Derived Rules

Modus Ponens

$$\frac{A \Rightarrow B \quad A}{B} \text{MP}$$

$$\frac{A \Rightarrow B \quad A \quad B}{B} \text{Imp E}$$

Left Conjunct

$$\frac{A \wedge B}{A} \text{AndL}$$

$$\frac{A \wedge B \quad A}{A} \text{And}_L \text{ E}$$

Right Conjunct

$$\frac{A \wedge B}{B} \text{AndR}$$

$$\frac{A \wedge B \quad B}{B} \text{And}_R \text{ E}$$

$$(A \wedge B) \Rightarrow (A \vee B)$$

Assumptions in Natural Deduction

- Problem: Keeping track of hypotheses and their discharge in Natural Deduction is *HARD!*
- Solution: Use *sequents* to track hypotheses
- A **sequent** is a pair of
 - A set of propositions (called assumptions, or hypotheses of sequent) and
 - A proposition (called conclusion of sequent)
- More generally (not here), allow set of hypotheses and set of conclusions

Nat. Ded. Introduction Sequent Rules

Γ is set of propositions (assumptions/hypotheses)

Hypothesis Introduction:

$$\frac{}{\Gamma \cup \{A\} \vdash A} \text{Hyp}$$

Truth Introduction:

$$\frac{}{\Gamma \vdash \mathbf{T}} \text{T I}$$

And Introduction:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \text{And I}$$

Or Introduction:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \text{Or}_L \text{ I}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \text{Or}_R \text{ I}$$

Not Introduction:

$$\frac{\Gamma \cup \{A\} \vdash \mathbf{F}}{\Gamma \vdash \neg A} \text{Not I}$$

Implication Introduction:

$$\frac{\Gamma \cup \{A\} \vdash B}{\Gamma \vdash A \Rightarrow B} \text{Imp I}$$

Nat. Ded. Elimination Sequent Rules

Γ is set of propositions (assumptions/hypotheses)

Not Elimination:

$$\frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash C} \text{Not E}$$

Implication Elimination:

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C} \text{Imp E}$$

And Elimination:

$$\frac{\Gamma \vdash A \wedge B \quad \Gamma \cup \{A\} \vdash C}{\Gamma \vdash C} \text{And}_L \text{ E}$$

$$\frac{\Gamma \vdash A \wedge B \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C} \text{And}_R \text{ E}$$

False Elimination:

$$\frac{\Gamma \vdash \mathbf{F}}{\Gamma \vdash C} \mathbf{F} \text{ E}$$

Or Elimination:

$$\frac{\Gamma \vdash A \vee B \quad \Gamma \cup \{A\} \vdash C \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C} \text{Or E}$$

Example Proof 4, Revisited

$$\{ \} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$$

Example Proof 4, Revisited

$$\frac{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{\{\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

Example Proof 4, Revisited

$$\frac{\frac{\frac{}{\{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I}}{\{\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

Example Proof 4, Revisited

$$\Gamma_3 = \{A \Rightarrow B, B \Rightarrow C, A\}$$

$$\frac{\frac{\frac{\{A \Rightarrow B, B \Rightarrow C, A\} \vdash C}{\{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C} \text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I}}{\{\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

Example Proof 4, Revisited

$$\Gamma_3 = \{A \Rightarrow B, B \Rightarrow C, A\}$$

$$\Gamma_4 = \{A \Rightarrow B, B \Rightarrow C, A, B\}$$

$$\frac{\frac{\frac{\frac{\frac{\Gamma_3 \vdash A \Rightarrow B \quad \Gamma_3 \vdash A}{\Gamma_4 \vdash C}}{\{A \Rightarrow B, B \Rightarrow C, A\} \vdash C}}{\{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}}{\{\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}}{\text{Imp E}}$$

Example Proof 4, Revisited

$$\Gamma_3 = \{A \Rightarrow B, B \Rightarrow C, A\}$$

$$\Gamma_4 = \{A \Rightarrow B, B \Rightarrow C, A, B\}$$

$$\begin{array}{c} \text{Hyp} \\ \hline \Gamma_3 \vdash A \Rightarrow B \quad \Gamma_3 \vdash A \quad \Gamma_4 \vdash C \\ \hline \Gamma_3 \vdash C \quad \text{Imp E} \\ \hline \{A \Rightarrow B, B \Rightarrow C, A\} \vdash C \\ \hline \{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C \quad \text{Imp I} \\ \hline \{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C) \quad \text{Imp I} \\ \hline \{\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)) \quad \text{Imp I} \end{array}$$

Example Proof 4, Revisited

$$\Gamma_3 = \{A \Rightarrow B, B \Rightarrow C, A\}$$

$$\Gamma_4 = \{A \Rightarrow B, B \Rightarrow C, A, B\}$$

$$\frac{\frac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \frac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \Gamma_4 \vdash C}{\frac{\frac{\frac{\frac{\{A \Rightarrow B, B \Rightarrow C, A\} \vdash C}{\{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C} \text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I}}{\{\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}} \text{Imp E}}$$

Example Proof 4, Revisited

$$\Gamma_3 = \{A \Rightarrow B, B \Rightarrow C, A\}$$

$$\Gamma_4 = \{A \Rightarrow B, B \Rightarrow C, A, B\}$$

$$\Gamma_5 = \{A \Rightarrow B, B \Rightarrow C, A, B, C\}$$

$$\frac{\frac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \frac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \frac{\frac{\Gamma_4 \vdash B \Rightarrow C}{\Gamma_4 \vdash B} \quad \Gamma_5 \vdash C}{\Gamma_4 \vdash C} \text{Imp E}}{\frac{\{A \Rightarrow B, B \Rightarrow C, A\} \vdash C}{\{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C} \text{Imp I}} \text{Imp E}$$
$$\frac{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{\{\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

Example Proof 4, Revisited

$$\Gamma_3 = \{A \Rightarrow B, B \Rightarrow C, A\}$$

$$\Gamma_4 = \{A \Rightarrow B, B \Rightarrow C, A, B\}$$

$$\Gamma_5 = \{A \Rightarrow B, B \Rightarrow C, A, B, C\}$$

$$\begin{array}{c}
 \text{Hyp} \quad \text{Hyp} \quad \text{Hyp} \quad \text{Hyp} \quad \text{Hyp} \\
 \hline
 \Gamma_3 \vdash A \Rightarrow B \quad \Gamma_3 \vdash A \quad \Gamma_4 \vdash B \Rightarrow C \quad \Gamma_4 \vdash B \quad \Gamma_5 \vdash C \\
 \hline
 \Gamma_4 \vdash C \quad \text{Imp E} \\
 \hline
 \{A \Rightarrow B, B \Rightarrow C, A\} \vdash C \quad \text{Imp E} \\
 \hline
 \{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C \quad \text{Imp I} \\
 \hline
 \{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C) \quad \text{Imp I} \\
 \hline
 \{\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)) \quad \text{Imp I}
 \end{array}$$

Example Proof 4, Revisited

$$\Gamma_3 = \{A \Rightarrow B, B \Rightarrow C, A\}$$

$$\Gamma_4 = \{A \Rightarrow B, B \Rightarrow C, A, B\}$$

$$\Gamma_5 = \{A \Rightarrow B, B \Rightarrow C, A, B, C\}$$

$$\frac{\frac{\frac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \frac{\text{Hyp}}{\Gamma_3 \vdash A}}{\Gamma_4 \vdash B \Rightarrow C} \text{Hyp} \quad \frac{\text{Hyp}}{\Gamma_4 \vdash B} \quad \frac{}{\Gamma_5 \vdash C}}{\Gamma_4 \vdash C} \text{Imp E}}{\frac{\frac{\frac{\frac{\frac{\frac{\{A \Rightarrow B, B \Rightarrow C, A\} \vdash C}{\{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C} \text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I}}{\{\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}}{\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp E}} \text{Imp E}$$

Example Proof 4, Revisited

$$\Gamma_3 = \{A \Rightarrow B, B \Rightarrow C, A\}$$

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$$\Gamma_5 = \{A \Rightarrow B, B \Rightarrow C, A, B, C\}$$

$$\begin{array}{c}
 \frac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \frac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \frac{\text{Hyp}}{\Gamma_4 \vdash B \Rightarrow C} \quad \frac{\text{Hyp}}{\Gamma_4 \vdash B} \quad \frac{\text{Hyp}}{\Gamma_5 \vdash C} \\
 \hline
 \frac{\Gamma_3 \vdash A \Rightarrow B \quad \Gamma_3 \vdash A \quad \Gamma_4 \vdash B \Rightarrow C \quad \Gamma_4 \vdash B \quad \Gamma_5 \vdash C}{\Gamma_4 \vdash C} \text{Imp E} \\
 \hline
 \frac{\{A \Rightarrow B, B \Rightarrow C, A\} \vdash C}{\{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C} \text{Imp I} \\
 \hline
 \frac{\{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)} \text{Imp I} \\
 \hline
 \frac{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{\{\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}
 \end{array}$$