CS477 Formal Software Dev Methods

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Slides based in part on previous lectures
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Example Tautology

$$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$$

A	B	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow B$	$A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$
true	false			

$$v = \{A \mapsto \mathit{True};\ B \mapsto \mathit{False}\}$$

$$\begin{aligned} \mathcal{I}_{\{A \mapsto \text{true};\ B \mapsto \text{false}\}}(A) &= \text{true} \\ \mathcal{I}_{\{A \mapsto \text{true};\ B \mapsto \text{false}\}}(B) &= \text{false} \end{aligned}$$

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Non-standard Model of Propositional Logic

Other models possible
Example:

- $\mathcal{C} = \{\text{true}, \text{false}, \bot\}$
- Valuations assign values in $\mathcal{C}$ to propositional atoms
- If $\mathcal{J}_w(p) = \bot$ then $\mathcal{J}_w(\neg p) = \bot$, otherwise same as for $\mathcal{I}$
- $\mathcal{J}_w(p) = \bot$ or $\mathcal{J}_w(q) = \bot$ then $\mathcal{J}_w(\neg p) = \bot$, $\mathcal{J}_w(p \wedge q) = \bot$, $\mathcal{J}_w(p \vee q) = \bot$, $\mathcal{J}_w(p \Rightarrow q) = \bot$, and $\mathcal{J}_w(p \Leftrightarrow q) = \bot$; otherwise same as for $\mathcal{I}$
- Note: $A \vee \neg A \not\equiv \mathbf{T}$
- Other variants possible

Proofs in Propositional Logic

- Natural Deduction proof is tree and a discharge function
  - Nodes are instances of inference rules
  - Leaves are assumptions of subproofs
  - Discharge function maps each leaf of the tree to an ancestor as allowed by the inference rules

Natural Deduction Inference Rules

- Inference rule has hypotheses and conclusion
- Conclusion a single proposition
- Hypotheses zero or more propositions, possibly with (discharged) hypotheses
- Rule with no hypotheses called an axiom
- Inference rule graphically presents as

$$\frac{\begin{array}{cccccccc} & A_i & & & & A_k & & \\ & \vdots & & & & \vdots & & \\ H_1 & \ldots & H_i & \ldots & H_j & \ldots & H_k & \ldots & H_n \end{array}}{C}\ rule$$

Natural Deduction Inference Rules

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- Two main kinds of inference rules:
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  - Eliminations – says how to use a proposition made from connective to prove result
    - Example:

$$\frac{A \Rightarrow B \quad A \quad \begin{array}{c} B \\ \vdots \\ C \end{array}}{C}\ \text{Imp E}$$

Introduction Rules

Truth Introduction:

$$\frac{}{\mathbf{T}}\ \mathbf{T}\ \mathrm{I}$$

And Introduction:

$$\frac{A \quad B}{A \wedge B}\ \mathrm{And\ I}$$

Or Introduction:

$$\frac{A}{A \vee B}\ \mathrm{Or}_L\ \mathrm{I} \qquad \frac{B}{A \vee B}\ \mathrm{Or}_R\ \mathrm{I}$$

Not Introduction:

$$\frac{\begin{array}{c}A\\ \vdots\\ \mathbf{F}\end{array}}{\neg A}\ \mathrm{Not\ I}$$

Implication Introduction:

$$\frac{\begin{array}{c}A\\ \vdots\\ B\end{array}}{A \Rightarrow B}\ \mathrm{Imp\ I}$$

No False Introduction

Example Proof 1

$$\frac{}{A \Rightarrow (B \Rightarrow (A \wedge B))}$$

Example Proof 1

$$\frac{\dfrac{\dfrac{A}{}}{B \Rightarrow (A \wedge B)}}{A \Rightarrow (B \Rightarrow (A \wedge B))}\ \mathrm{Imp\ I}$$

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$$\frac{\dfrac{\dfrac{A \quad B}{A \wedge B}}{B \Rightarrow (A \wedge B)}\ \mathrm{Imp\ I}}{A \Rightarrow (B \Rightarrow (A \wedge B))}\ \mathrm{Imp\ I}$$

Example Proof 1

$$\frac{\dfrac{\dfrac{A \quad B}{A \wedge B}\ \mathrm{And\ I}}{B \Rightarrow (A \wedge B)}\ \mathrm{Imp\ I}}{A \Rightarrow (B \Rightarrow (A \wedge B))}\ \mathrm{Imp\ I}$$

Example Proof 1

$$\frac{\dfrac{\dfrac{A \quad B}{A \wedge B}\ \mathrm{And\ I}}{B \Rightarrow (A \wedge B)}\ \mathrm{Imp\ I}}{A \Rightarrow (B \Rightarrow (A \wedge B))}\ \mathrm{Imp\ I}$$

- All assumptions discharged; proof complete

Example Proof 2

$$\frac{}{B \Rightarrow (A \wedge B)}$$

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$$\frac{\dfrac{\dfrac{}{B}}{A \wedge B}}{B \Rightarrow (A \wedge B)}\ \text{Imp I}$$

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$$\frac{\dfrac{\dfrac{A \quad B}{A \wedge B}\ \text{And I}}{B \Rightarrow (A \wedge B)}\ \text{Imp I}$$

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- Closed proofs must discharge all hypotheses
- Otherwise have theorem relative to / under undischarged hypotheses
- Here have proved "Assuming A, we have $B \Rightarrow (A \wedge B)$

Discharging Hypothesis

$$\frac{}{A \Rightarrow (A \wedge A)}$$

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- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\frac{\dfrac{A \quad A}{A \wedge A}\,\text{And I}}{A \Rightarrow (A \wedge A)}\,\text{Imp I} \qquad \frac{}{A \Rightarrow (B \Rightarrow A)}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\frac{\dfrac{A \quad A}{A \wedge A}\,\text{And I}}{A \Rightarrow (A \wedge A)}\,\text{Imp I} \qquad \frac{\dfrac{A}{B \Rightarrow A}\,\text{Imp I}}{A \Rightarrow (B \Rightarrow A)}\,\text{Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis

Discharging Hypothesis

$$\frac{\dfrac{A \quad A}{A \wedge A}\,\text{And I}}{A \Rightarrow (A \wedge A)}\,\text{Imp I} \qquad \frac{\dfrac{A}{B \Rightarrow A}\,\text{Imp I}}{A \Rightarrow (B \Rightarrow A)}\,\text{Imp I}$$

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Discharging Hypothesis

$$\dfrac{\dfrac{A \quad A}{A \wedge A}\,\text{And I}}{A \Rightarrow (A \wedge A)}\,\text{Imp I} \qquad \dfrac{\dfrac{A}{B \Rightarrow A}\,\text{Imp I}}{A \Rightarrow (B \Rightarrow A)}\,\text{Imp I}$$

- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis
- Or may discharge none at all
- Every assumption instance discharged only once

Your Turn

$$\dfrac{}{A \Rightarrow (A \vee B)}$$

Elimination Rules

- So far, have rules to "introduce" logical connectives into propositions
- No rules for how to "use" logical connectives
  - No assumptions with logical connectives
- Need "elimination" rules
- Example: Can't prove

$$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$$

  with what we have so far
- Elimination rules assume assumption with a connective; have general conclusion
  - Generally needs additional hypotheses

Elimination Rules

False Elimination:

$$\dfrac{\mathbf{F}}{C}\,\mathbf{F}\ \text{E}$$

Not Elimination:

$$\dfrac{\neg A \quad A}{C}\,\text{Not E}$$

And Elimination:

$$\dfrac{\begin{array}{c}A\\ \vdots\\ A \wedge B \quad C\end{array}}{C}\,\text{And}_L\ \text{E} \qquad \dfrac{\begin{array}{c}B\\ \vdots\\ A \wedge B \quad C\end{array}}{C}\,\text{And}_R\ \text{E}$$

Or Elimination:

$$\dfrac{\begin{array}{ccc}A & B\\ \vdots & \vdots\\ A \vee B \quad C & C\end{array}}{C}\,\text{Or E}$$

Implication Elimination:

$$\dfrac{\begin{array}{ccc} & B\\ & \vdots\\ A \Rightarrow B \quad A & C\end{array}}{C}\,\text{Imp E}$$

Example Proof 4

$$\dfrac{}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}$$

Example Proof 4

$$\dfrac{\dfrac{}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$

Example Proof 4

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow C}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\,\text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$

Example Proof 4

$$\dfrac{\dfrac{\dfrac{\dfrac{}{C}}{A \Rightarrow C}\,\text{Imp I}}{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\,\text{Imp I}}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}$$

Example Proof 4

$$\dfrac{\dfrac{\dfrac{A \Rightarrow B \quad A \quad \dfrac{}{C}}{C}\,\text{Imp E}}{A \Rightarrow C}\,\text{Imp I}}{\dfrac{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}}\,\text{Imp I}$$

Example Proof 4

$$\dfrac{\dfrac{\dfrac{A \Rightarrow B \quad A \quad \dfrac{}{C}}{C}\,\text{Imp E}}{A \Rightarrow C}\,\text{Imp I}}{\dfrac{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}}\,\text{Imp I}$$

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$$\dfrac{\dfrac{\dfrac{A \Rightarrow B \quad A \quad \dfrac{B \Rightarrow C \quad B \quad C}{C}\,\text{Imp E}}{C}\,\text{Imp E}}{A \Rightarrow C}\,\text{Imp I}}{\dfrac{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\,\text{Imp I}}\,\text{Imp I}$$

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Some Well-Known Derived Rules

Modus Ponens

$$\dfrac{A \Rightarrow B \quad A}{B}\,\text{MP} \qquad \dfrac{A \Rightarrow B \quad A \quad B}{B}\,\text{Imp E}$$

Left Conjunct

$$\dfrac{A \wedge B}{A}\,\text{AndL} \qquad \dfrac{A \wedge B \quad A}{A}\,\text{And}_L\text{ E}$$

Right Conjunct

$$\dfrac{A \wedge B}{B}\,\text{AndR} \qquad \dfrac{A \wedge B \quad B}{B}\,\text{And}_R\text{ E}$$

Your Turn

$$\dfrac{}{(A \wedge B) \Rightarrow (A \vee B)}$$

Assumptions in Natural Deduction

- Problem: Keeping track of hypotheses and their discharge in Natural Deduction is *HARD!*
- Solution: Use *sequents* to track hypotheses
- A sequent is a pair of
  - A set of propositions (called assumptions, or hypotheses of sequent) and
  - A proposition (called conclusion of sequent)
- More generally (not here), allow set of hypotheses and set of conclusions

Nat. Ded. Introduction Sequent Rules

Γ is set of propositions (assumptions/hypotheses)
Hypothesis Introduction:

$$\frac{}{\Gamma \cup \{A\} \vdash A}\ \text{Hyp}$$

Truth Introduction:

$$\frac{}{\Gamma \vdash \mathbf{T}}\ \mathbf{T}\ \text{I}$$

And Introduction:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \text{And I}$$

Or Introduction:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \text{Or}_L\ \text{I} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \text{Or}_R\ \text{I}$$

Not Introduction:

$$\frac{\Gamma \cup \{A\} \vdash \mathbf{F}}{\Gamma \vdash \neg A}\ \text{Not I}$$

Implication Introduction:

$$\frac{\Gamma \cup \{A\} \vdash B}{\Gamma \vdash A \Rightarrow B}\ \text{Imp I}$$

Nat. Ded. Elimination Sequent Rules

Γ is set of propositions (assumptions/hypotheses)
Not Elimination:

$$\frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash C}\ \text{Not E}$$

Implication Elimination:

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C}\ \text{Imp E}$$

And Elimination:

$$\frac{\Gamma \vdash A \wedge B \quad \Gamma \cup \{A\} \vdash C}{\Gamma \vdash C}\ \text{And}_L\ \text{E} \qquad \frac{\Gamma \vdash A \wedge B \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C}\ \text{And}_R\ \text{E}$$

False Elimination:

$$\frac{\Gamma \vdash \mathbf{F}}{\Gamma \vdash C}\ \mathbf{F}\ \text{E}$$

Or Elimination:

$$\frac{\Gamma \vdash A \vee B \quad \Gamma \cup \{A\} \vdash C \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C}\ \text{Or E}$$

Example Proof 4, Revisited

$$\frac{}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}$$

Example Proof 4, Revisited

$$\frac{\dfrac{}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\ \text{Imp I}}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\ \text{Imp I}$$

Example Proof 4, Revisited

$$\frac{\dfrac{\dfrac{}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\ \text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\ \text{Imp I}}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\ \text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$

$$\frac{\dfrac{\dfrac{\dfrac{}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\ \text{Imp I}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\ \text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\ \text{Imp I}}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\ \text{Imp I}$$

Example Proof 4, Revisited

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$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\Gamma_3 \vdash A \Rightarrow B} \quad \overline{\Gamma_3 \vdash A} \quad \overline{\Gamma_4 \vdash C}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\text{Imp I}}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \overline{\Gamma_3 \vdash A} \quad \overline{\Gamma_4 \vdash C}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\text{Imp I}}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\text{Imp I}$$

Example Proof 4, Revisited

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \dfrac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \overline{\Gamma_4 \vdash C}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\text{Imp I}}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
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$\Gamma_5 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B,\ C\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \dfrac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \dfrac{\overline{\Gamma_4 \vdash B \Rightarrow C} \quad \overline{\Gamma_4 \vdash B} \quad \overline{\Gamma_5 \vdash C}}{\Gamma_4 \vdash C}\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\text{Imp I}}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\text{Imp I}$$

Example Proof 4, Revisited

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Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
$\Gamma_4 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B\}$
$\Gamma_5 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B,\ C\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Hyp}}{\Gamma_3 \vdash A \Rightarrow B} \quad \dfrac{\text{Hyp}}{\Gamma_3 \vdash A} \quad \dfrac{\dfrac{\text{Hyp}}{\Gamma_4 \vdash B \Rightarrow C} \quad \dfrac{\text{Hyp}}{\Gamma_4 \vdash B} \quad \overline{\Gamma_5 \vdash C}}{\Gamma_4 \vdash C}\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\text{Imp I}}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\text{Imp I}$$

Example Proof 4, Revisited

$\Gamma_3 = \{A \Rightarrow B,\ B \Rightarrow C,\ A\}$
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$\Gamma_5 = \{A \Rightarrow B,\ B \Rightarrow C,\ A,\ B,\ C\}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_3 \vdash A \Rightarrow B}\text{Hyp} \quad \dfrac{}{\Gamma_3 \vdash A}\text{Hyp} \quad \dfrac{}{\Gamma_4 \vdash B \Rightarrow C}\text{Hyp} \quad \dfrac{}{\Gamma_4 \vdash B}\text{Hyp} \quad \dfrac{}{\Gamma_5 \vdash C}\text{Hyp}}{\Gamma_4 \vdash C}\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C,\ A\} \vdash C}\text{Imp E}}{\{A \Rightarrow B,\ B \Rightarrow C\} \vdash A \Rightarrow C}\text{Imp I}}{\{A \Rightarrow B\} \vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\text{Imp I}}{\{\,\} \vdash (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))}\text{Imp I}$$